

# On the Categorification of the Möbius Function

Rafael Díaz

## Abstract

In these notes we study several categorical generalizations of the Möbius function and discuss the relations between the various approaches. We emphasize the topological and geometric meaning of these constructions.

## 1 Introduction

The Möbius function  $\mu : \mathbb{N}_+ \rightarrow \mathbb{Z}$  is the map from the positive natural numbers to the integers such that  $\mu(1) = 1$ ,  $\mu(n) = 0$  if  $n$  has repeated prime factors, and  $\mu(n) = (-1)^k$  if  $n$  is the product of  $k$  distinct prime numbers. It was introduced by Möbius in 1832 and it has been since an important tool in number theory, complex analysis, and combinatorics. Our goal in these notes is to gently introduced several generalizations of the Möbius function originating from the viewpoint of category theory, which have emerged thanks to the contribution of various authors.

We begin with a review of the classical Möbius theory where the main character is the set of the positive natural numbers  $\mathbb{N}_+$ , which for our purposes can be regarded either as a poset or as a monoid. We adopt first the poset viewpoint. It leads to the construction of Möbius theories for locally finite posets (Rota), for locally finite directed graphs, for locally finite categories (Haigh, Leinster), and for essentially locally finite categories. Then we adopt the monoid viewpoint, which leads to the development of Möbius theories for finite decomposition monoids, for Möbius categories (Haigh, Leroux), and for essentially finite decomposition categories.

Once we have developed Möbius theories for a variety of mathematical objects, a couple of natural question arise. How the various theories relate to each other? We will address this question as we develop the various theories along the notes. As a rule these relations are quite subtle a non fully functorial. For example, one may wonder how the relation between the category of posets and the category of finite decomposition monoids may be characterized.

The second question asks for the common features among Möbius theories. In the examples we have found the following structural features:

- The theory depends on a fixed ring  $R$  and applies to some category of objects  $C$ . For

simplicity in these notes we consider the characteristic zero case. In a fully categorified approach one may even want to let  $R$  be a ring-like category.

- There is a construction that associates a  $R$ -algebra  $A_c$  to each object  $c \in C$ , called the incidence or convolution algebra of  $c$ . The correspondence  $c \mapsto A_c$  need not be functorial. However, it is functorial under isomorphism, i.e. we have a functor

$$A : C_g \longrightarrow R\text{-alg}$$

from the underlying groupoid of  $C$  to the category of  $R$ -algebras. Often the category  $C$  comes equipped with a notion of weak equivalences, which are a suitable family of morphisms between objects of  $C$ . The construction may fail to be functorial under equivalences even if it is functorial under isomorphism.

- The category  $C$  is monoidal with product  $\times : C \times C \longrightarrow C$ . The category of  $R$ -algebras is monoidal with product the (suitable completed) tensor product  $\otimes$ . The functor

$$A : C_g \longrightarrow R\text{-alg}$$

is monoidal, i.e. there are natural isomorphisms

$$A_{c \times d} \simeq A_c \otimes A_d.$$

- There is a simple characterization of the units (invertible elements) in  $A_c$ . Moreover, there is a terminating recursive procedure that finds the inverse  $u^{-1}$  for each unit  $u$  of  $A_c$ . There are some distinguished unit elements, say  $\xi_c \in A_c$ , usually with a straightforward definition, such that their inverses  $\mu_c$  are unexpectedly useful having interesting topological and geometrical properties.
- Often each algebra  $A_c$  comes equipped with a natural  $R$ -module  $M_c$ , leading to the Möbius inversion formula:

$$a = b\xi_c \quad \text{if and only if} \quad b = a\mu_c \quad \text{for all} \quad a, b \in M_c.$$

We proceed to develop our examples of Möbius theories, providing along the notes references to some of the main contributions in the field, and discussing the relation between the various theories. We focus on the topological approach to understand the meaning of the Möbius functions, in particular, the Euler characteristics for simplicial groupoids will be useful for us. We emphasize that there is not a unique categorical generalization for the Möbius functions, but rather several approaches, each with its own advantages and applications. Although we explore various routes, this work is not meant to be an exhaustive study. That would be a task exceeding the limits of these short notes. For example, we do not cover the advances developed by

Cartier and Foata [9], Dür [12], Fiore, Lück and Sauer [13], among others.

This notes were prepared for the CIMPA school "Modern Methods in Combinatorics," San Luis, Argentina 2013. For the reader convenience we include a few exercises.

## 2 Classical Möbius Theory

In this section we review the basic elements of the classical Möbius theory [2]. Let  $\mathbb{N}_+$  be the set of positive natural numbers and  $R$  be a commutative ring with identity.

**Definition 1.** The Möbius function is the map  $\mu : \mathbb{N}_+ \longrightarrow R$  given by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ has repeated prime factors,} \\ (-1)^k & \text{if } n = p_1 \dots p_k, \text{ with } p_j \text{ distinct prime numbers.} \end{cases}$$

Let  $[\mathbb{N}_+, R]$  be the set of maps from  $\mathbb{N}_+$  to  $R$ . We write  $d|n$  if  $d$  divides  $n$ .

**Definition 2.** The Dirichlet product  $\star$  on  $[\mathbb{N}_+, R]$  is given on maps  $f, g \in [\mathbb{N}_+, R]$  by:

$$f \star g(n) = \sum_{d|n} f(d)g(n/d) = \sum_{d|n} f(n/d)g(d) = \sum_{cd=n} f(c)g(d).$$

The unit for  $\star$  is the map  $1 : \mathbb{N}_+ \longrightarrow \mathbb{Z}$  given by  $1(1) = 1$  and  $1(n) = 0$  for  $n \geq 2$ .

The product  $\star$  turns  $[\mathbb{N}_+, R]$  into a commutative  $R$ -algebra. Next result is known as the Möbius inversion formula.

### Proposition 3. (Möbius Inversion Formula)

Let  $\xi \in [\mathbb{N}_+, R]$  be the map constantly equal to 1. The Möbius function  $\mu \in [\mathbb{N}_+, R]$  is the  $\star$ -inverse of  $\xi$ . Thus for  $f \in [\mathbb{N}_+, R]$ , we have that

$$g = f \star \xi \quad \text{if and only if} \quad f = g \star \mu.$$

Equivalently,

$$g(n) = \sum_{d|n} f(d) \quad \text{if and only if} \quad f(n) = \sum_{d|n} \mu(n/d)g(d).$$

*Proof.* We show that  $\xi \star \mu = 1$ . Since

$$\xi \star \mu(1) = \xi(1)\mu(1) = 1,$$

it only remains to check that  $\xi \star \mu(n) = 0$  for  $n \geq 2$ . Given  $n = p_1^{a_1} \dots p_k^{a_k}$ , with  $p_j$  distinct prime numbers, we have that

$$\xi \star \mu(p_1^{a_1} \dots p_k^{a_k}) = \sum_{d|p_1^{a_1} \dots p_k^{a_k}} \mu(d) = 1 + \sum_{l=1}^k \sum_{A \subseteq [k], |A|=l} \mu\left(\prod_{i \in A} p_i\right) =$$

$$1 + \sum_{l=1}^k \sum_{A \subseteq [k], |A|=l} (-1)^l = \sum_{l=0}^k (-1)^l \binom{k}{l} = (1-1)^k = 0.$$

□

**Exercise 4.** Show that  $f \in [\mathbb{N}_+, R]$  is a  $\star$ -unit if and only if  $f(1)$  is a unit in  $R$ .

**Exercise 5.** A map  $f \in [\mathbb{N}_+, R]$  is multiplicative if  $f(ab) = f(a)f(b)$  for  $a, b$  coprime. Show that  $1, \xi$  and  $\mu$  are multiplicative functions. Show that  $f \star g$  is multiplicative if  $f$  and  $g$  are multiplicative. Show that the  $\star$ -inverse of a multiplicative function is multiplicative.

Rooted in number theory the Möbius function also plays an important role in complex analysis through the theory of Dirichlet series.

**Definition 6.** A Dirichlet series with coefficients in  $R$  is a formal power series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \text{where } a_n \in R.$$

We let  $\mathbb{D}_R$  be the  $R$ -algebra of Dirichlet series with the  $R$ -linear product given by

$$\frac{1}{n^s} \frac{1}{m^s} = \frac{1}{(nm)^s}.$$

For our present purposes we may regard  $s$  as a formal variable. In most applications one usually works over the complex numbers  $\mathbb{C}$ , the variable  $s$  is  $\mathbb{C}$ -valued, and convergency issues are of the greatest importance. The product of convergent Dirichlet series is just the product of complex valued functions.

The Dirichlet algebra  $\mathbb{D}_R$  is isomorphic to the  $R$ -algebra  $R \langle\langle x_1, \dots, x_n, \dots \rangle\rangle$  of formal  $R$ -linear combinations in the variables  $x_n$ , with the  $R$ -linear product given by

$$x_n x_m = x_{nm}, \quad \text{via the map sending } x_n \text{ to } \frac{1}{n^s}.$$

Note that if  $n = p_1^{a_1} \dots p_k^{a_k}$ , then we have that  $x_n = x_{p_1}^{a_1} \dots x_{p_k}^{a_k}$ . Therefore  $\mathbb{D}_R$  is isomorphic to the algebra  $R[[x_2, x_3, x_5, \dots]]$  of formal power series with coefficients in  $R$  in the variables  $x_p$ , with  $p$  a prime number.

**Proposition 7.** The map  $D : [\mathbb{N}_+, R] \longrightarrow \mathbb{D}_R$  sending  $f \in [\mathbb{N}_+, R]$  to the Dirichlet series

$$D_f = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{is a ring isomorphism.}$$

For example, the Dirichlet series  $D_\xi$  associated with  $\xi$  is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{and we have that} \quad \zeta^{-1}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

**Exercise 8.** Describe explicitly the product of Dirichlet series and prove Proposition 7.

**Exercise 9.** For  $p \geq 2$  a primer number, let  $f_p \in [\mathbb{N}_+, \mathbb{Z}]$  be such that  $f_p(n) = 1$  if  $n = p^l$  for some  $l \in \mathbb{N}$ , and  $f_p(n) = 0$  otherwise. Find  $f_p^{-1}$ ,  $D_{f_p}$  and  $D_{f_p}^{-1}$ .

**Exercise 10.** Let  $\eta \in [\mathbb{N}_+, \mathbb{Z}]$  be such that  $\eta(1) = 1$ ,  $\eta(p) = -1$  for  $p$  a prime number, and  $\eta(n) = 0$  otherwise. Find  $\eta^{-1}$ ,  $D_\eta$  and  $D_\eta^{-1}$ .

We close this section with a couple of facts (kind of beyond the main topic of these notes) that show the relevance of the Möbius function in number theory and complex analysis. The first fact is a reformulation in terms of the Möbius function of a main theorem in number theory on the asymptotic behaviour of prime numbers [2].

**Theorem 11. (Prime Number Theorem and the Möbius Function)**

Let  $\pi : \mathbb{N}_+ \rightarrow \mathbb{N}$  be such that  $\pi(n)$  counts the prime numbers less than or equal to  $n$ . We have that:

$$\lim_{n \rightarrow \infty} \frac{\pi(n) \ln(n)}{n} = 1, \quad \text{or equivalently,} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k \leq n} \mu(k) = 0.$$

The second fact relates the Möbius function to the Riemann Hypothesis [30, 32]. The Riemann zeta function  $\zeta(s)$  can be analytically extended to the complex plane with a pole at  $s = 1$ . The  $\zeta$  function has zeroes at  $s = -2, -4, -6, \dots$ , the so called trivial zeroes. Any other zero is called nontrivial, and the Riemann Hypothesis claims that all of them have real part equal to  $\frac{1}{2}$ .

**Theorem 12. (Riemann Hypothesis and the Möbius Function)**

The real part of the non-trivial zeroes of the Riemann zeta function is  $\frac{1}{2}$  if and only if for any  $\epsilon > 0$  there exists  $M > 0$  such that

$$\sum_{k \leq n} \mu(k) < M n^{\frac{1}{2} + \epsilon}.$$

### 3 Locally Finite Posets

In the 60's a comprehensive Möbius theory for locally finite partially ordered sets (posets) was developed by Gian-Carlo Rota and his collaborators, among them Crapo, Stanley, Schmitt. Posets are actually a particular kind of categories, so this early development may already be regarded as a categorification of the classical Möbius theory. For a classical introduction to categories the reader may consult Mac Lanes's book [26]. Another, quite readable, introduction

to the subject is provided by Lawvere and Schanuel [21].

Almost all the results in this section can be found in the first chapter of [28], where the reader will find a nice mixture of original and review papers. The newer results in this section are a discrete analogue for the Gauss-Bonnet theorem for finite posets, and the Leinster's characterization of the relation between Möbius and matrix inversion for finite posets [22, 23].

A partial order on a set  $X$  is a reflexive, antisymmetric, and transitive relation on  $X$ . A poset (partially ordered set) is a pair  $(X, \leq)$  where  $X$  is a set, and  $\leq$  is a partial order on  $X$ . Morphism between posets are order preserving maps. For  $x \leq y$  in  $X$ , the interval  $[x, y]$  is the subset of  $X$  given by

$$[x, y] = \{z \in X \mid x \leq z \leq y\}.$$

We let

$$\mathbb{I}_X = \{[x, y] \mid x, y \in X, \ x \leq y\}$$

be the set of all intervals of  $X$ .

**Definition 13.** A poset is locally finite if its intervals are finite sets.

We need a few algebraic notions such as coalgebras and Hopf algebras. The reader may consult Kassel's book [17] for a comprehensive introduction to these notions in the context quantum groups, and Cartier's notes [8] for an historical introduction highlighting the topological connection.

**Lemma 14.** Let  $X$  be a locally finite poset. The free  $R$ -module  $\langle \mathbb{I}_X \rangle$  generated by  $\mathbb{I}_X$  together with the  $R$ -linear maps

$$\Delta : \langle \mathbb{I}_X \rangle \longrightarrow \langle \mathbb{I}_X \rangle \otimes \langle \mathbb{I}_X \rangle \quad \text{and} \quad \epsilon : \langle \mathbb{I}_X \rangle \longrightarrow R$$

given on generators, respectively, by

$$\Delta[x, z] = \sum_{x \leq y \leq z} [x, y] \otimes [y, z] \quad \text{and} \quad \epsilon[x, y] = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases}$$

is a  $R$ -coalgebra.

*Proof.* The counit property is clear, indeed we have that

$$(\epsilon \otimes 1)\Delta[x, y] = \epsilon[x, x][x, y] = [x, y] = [x, y]\epsilon[y, y] = (1 \otimes \epsilon)\Delta[x, y].$$

Coassociativity follows from the identities

$$(\Delta \otimes 1)\Delta[x, w] = \sum_{x \leq y \leq z \leq w} [x, y] \otimes [y, z] \otimes [z, w] = (1 \otimes \Delta)\Delta[x, w].$$

□

Recall that if  $C$  is a  $R$ -coalgebra, then the dual  $R$ -module

$$C^* = \text{Hom}_R(C, R)$$

is an  $R$ -algebra with product  $\star : C^* \otimes C^* \longrightarrow C^*$  given on  $f, g \in C^*$  by

$$f \star g(c) = (f \otimes g)\Delta(c).$$

The  $\star$ -unit of  $C^*$  is the counit  $\epsilon$  of  $C$ .

For the coalgebra  $\langle \mathbb{I}_X \rangle$  of intervals, the dual algebra  $\langle \mathbb{I}_X \rangle^*$  may be identified with the algebra  $[\mathbb{I}_X, R]$  of maps from  $\mathbb{I}_X$  to  $R$  with the product given by

$$(f \star g)[x, z] = \sum_{x \leq y \leq z} f[x, y]g[y, z].$$

The  $R$ -algebra  $([\mathbb{I}_X, R], \star)$  is called the incidence algebra of  $X$ .

**Theorem 15.** Let  $(X, \leq)$  be a locally finite poset. A map  $f \in [\mathbb{I}_X, R]$  is invertible in the incidence algebra if and only if  $f[x, x]$  is a unit for all  $x \in X$ . In particular, the incidence map  $\xi \in [\mathbb{I}_X, R]$  constantly equal to 1 is invertible; its inverse  $\mu \in [\mathbb{I}_X, R]$  is called the Möbius function of the poset  $(X, \leq)$ .

*Proof.* If  $f \star g = \epsilon$ , then  $f[x, x]g[x, x] = 1$  and thus necessarily  $f[x, x]$  is a unit for all  $x \in X$ . Conversely, assume that  $f[x, x]$  is a unit for  $x \in X$ , then its inverse  $g$ , if it exists, must satisfy  $g[x, x] = f[x, x]^{-1}$  for  $x \in X$ , and for  $x < z$  in  $X$  we have that

$$f[x, x]g[x, z] + \sum_{x < y \leq z} f[x, y]g[y, z] = 0,$$

or equivalently

$$g[x, z] = - \sum_{x < y \leq z} \frac{f[x, y]g[y, z]}{f[x, x]}.$$

The equations above give a terminating recursive definition for  $g$  since  $X$  is locally finite. Solving the recursion we get for  $x < y$  in  $X$  that

$$g[x, y] = \sum_{n \geq 1} \sum_{x=x_0 < x_1 < \dots < x_n=y} (-1)^n \frac{f[x_0, x_1] \dots f[x_{n-1}, x_n]}{f[x_0, x_0] f[x_1, x_1] \dots f[x_{n-1}, x_{n-1}] f[x_n, x_n]}$$

□

**Corollary 16.** The Möbius function  $\mu \in [\mathbb{I}_X, R]$  of a locally finite poset  $(X, \leq)$  is given by  $\mu[x, x] = 1$  for  $x \in X$ , and for  $x < y$  in  $X$  it is given by

$$\mu[x, y] = \sum_{n \geq 1} (-1)^n |\{x = x_0 < x_1 < \dots < x_{n-1} < x_n = y\}|.$$

**Proposition 17.** Let  $(X, \leq)$  and  $(Y, \leq)$  be finite posets. Give  $X \times Y$  the poset structure

$$(a_1, a_2) \leq (b_1, b_2) \quad \text{if and only if} \quad a_1 \leq b_1 \quad \text{and} \quad a_2 \leq b_2.$$

There is natural bijection  $\mathbb{I}_{X \times Y} \longrightarrow \mathbb{I}_X \times \mathbb{I}_Y$  sending  $[(a_1, a_2), (b_1, b_2)]$  to  $([a_1, b_1], [a_2, b_2])$  inducing an isomorphism

$$([\mathbb{I}_{X \times Y}, R], \star) \simeq ([\mathbb{I}_X, R], \star) \otimes ([\mathbb{I}_Y, R], \star).$$

Moreover, we have that  $\xi_{X \times Y}[(a_1, a_2), (b_1, b_2)] = \xi_X[a_1, b_1]\xi_Y[a_2, b_2]$  and thus

$$\mu_{X \times Y}[(a_1, a_2), (b_1, b_2)] = \mu_X[a_1, b_1]\mu_Y[a_2, b_2].$$

**Remark 18.** For infinite posets we have a natural embedding of algebras

$$([\mathbb{I}_X, R], \star) \otimes ([\mathbb{I}_Y, R], \star) \longrightarrow ([\mathbb{I}_{X \times Y}, R], \star),$$

which can be promoted to an isomorphisms by using a suitable completed version of the tensor product.

Next exercises compute the Möbius function for a few families of posets. The reader may consult [4] for these and more examples.

**Exercise 19.** For  $n \leq m$  in  $\mathbb{N}$  show that the Möbius function of the interval  $[n, m] = \{n, \dots, m\}$  is given by  $\mu[n, n] = 1$ ,  $\mu[n, n+1] = -1$ , and  $\mu[n, m] = 0$ , for  $m \geq n+2$ .

**Exercise 20.** For a finite set  $X$ , let  $(PX, \subseteq)$  be the set of subsets of  $X$  ordered by inclusion. Show that the Möbius function of  $(PX, \subseteq)$  is given for  $A \subseteq B$  by

$$\mu[A, B] = (-1)^{|A \setminus B|}$$

**Exercise 21.** Given a finite set  $X$ , let  $(\text{Par}X, \leq)$  be set of partitions of  $X$ , and for  $\pi, \sigma \in \text{Par}X$  we set  $\pi \leq \sigma$  if and only if each block of  $\pi$  is included in a block of  $\sigma$ . Show that the Möbius function of  $(\text{Par}X, \leq)$  is given by

$$\mu[\pi, \sigma] = (-1)^{|\pi| - |\sigma|} \prod_{b \in \sigma} (n_b - 1)!,$$

where  $n_b$  is the number of blocks of  $\pi$  included in  $b$ .

Our next results require a few notions in combinatorial algebraic topology. Kozlov in [19] provides a self-contained introduction to the subject, and in particular discusses homology theory in Chapter 3.

**Definition 22.** A simplicial complex  $C$  consists of a set of vertices  $X$  together with a family  $C$  of finite non-empty subsets of  $X$ , called simplices, such that:



- $\{x\} \in C$  for all  $x \in X$ .
- If  $\emptyset \neq A \subseteq B$  and  $B \in C$ , then  $A \in C$ .

To a finite poset  $(X, \leq)$  we associate the simplicial complex  $CX$  of linearly ordered subsets of  $X$ . It has vertex set  $X$  and is such that  $c \subseteq X$  is in  $CX$  if and only if the restriction of  $\leq$  to  $c$  is a linear, or total, order.

For  $n \geq -1$ , we let  $C_nX$  be the set of elements in  $CX$  of cardinality  $n+1$ . By convention we let the empty set be the unique element of  $C_{-1}X$ .

The geometric realization of  $CX$  is the topological space

$$|CX| = \left\{ a \in [X, [0, 1]] \mid \text{support}(a) \in C \text{ and } \sum_{x \in X} a(x) = 1 \right\},$$

where

- $\text{support}(a) = \{x \in X \mid a(x) \neq 0\}$ , and  $[0, 1]$  is the unit interval in  $\mathbb{R}$ ,
- the space  $[X, [0, 1]] = \prod_{x \in X} [0, 1]$  is given the product topology,
- $|CX| \subseteq [X, [0, 1]]$  is given the subspace topology.

The reduced homology groups  $\tilde{H}_n|CX|$  of  $|CX|$ , for  $n \geq -1$ , may be identified with the homology groups of the differential complex  $(\langle CX \rangle, d)$  such that

$$\langle CX \rangle = \bigoplus_{n \geq -1} \langle C_nX \rangle,$$

where  $\langle C_nX \rangle$  is the  $\mathbb{Z}$ -module generated by  $C_nX$ . The differential map

$$d : \langle C_nX \rangle \longrightarrow \langle C_{n-1}X \rangle$$

is given on generators by

$$d\{x_0 < \dots < x_n\} = \sum_{i=0}^n (-1)^i \{x_0 < \dots < \hat{x}_i < \dots < x_n\}.$$

Therefore the reduced Euler characteristic of  $|CX|$  is given by

$$\tilde{\chi}|CX| = \sum_{n \geq -1} (-1)^n |C_nX| = \sum_{n \geq 0} (-1)^n \text{rank} \tilde{H}_n|CX|.$$

The left identity holds by definition, the right identity is proven in [19]. We recall that the rank of an abelian group counts the number of generators of the free part of the group.

The Euler characteristic of  $|CX|$  is defined in terms of the homology groups of the differential complex

$$(< C_{\geq 0}X >, d) \subseteq (< CX >, d),$$

where

$$< C_{\geq 0}X > = \bigoplus_{n \geq 0} < C_n X >, \quad \text{as follows}$$

$$\chi|CX| = \sum_{n \geq 0} (-1)^k |C_n X| = \sum_{n \geq 0} (-1)^n \text{rank} H_n(|CX|).$$

The following topological results are due to P. Hall.

**Theorem 23. (Homological Interpretation of the Möbius Function)**

The Möbius function  $\mu \in [\mathbb{I}_X, R]$  of a locally finite poset  $(X, \leq)$  is given for  $x < y \in X$  by

$$\mu[x, y] = \tilde{\chi}|C(x, y)| = \sum_{n \geq -1} (-1)^n |C_n(x, y)| = \sum_{n \geq 0} (-1)^n \text{rank} \tilde{H}_n |C(x, y)|,$$

where  $(x, y)$  is the interval  $\{z \mid x < z < y\} \subseteq X$  with the induced order.

*Proof.*

$$\begin{aligned} \mu[x, y] &= \sum_{n \geq 1} (-1)^n |\{x = x_0 < x_1 < \dots < x_n = y\}| = \\ &= \sum_{n \geq 1} (-1)^{n-2} |C_{n-2}(x, y)| = \sum_{n \geq -1} (-1)^n |C_n(x, y)| = \tilde{\chi}|C(x, y)|. \end{aligned}$$

□

Let  $X$  be a finite poset, and  $\overline{X}$  be the poset obtained from  $X$  by adjoining a minimum  $\widehat{0}$  and a maximum  $\widehat{1}$  to  $X$ .

**Corollary 24.** Let  $\mu_{\overline{X}} \in [\mathbb{I}_{\overline{X}}, R]$  be the Möbius function of  $\overline{X}$ , then

$$\mu_{\overline{X}}[\widehat{0}, \widehat{1}] = \tilde{\chi}|CX| = \sum_{n \geq -1} (-1)^n \text{rank} \tilde{H}_n |CX|.$$

*Proof.* Follows from Theorem 23 since  $(\widehat{0}, \widehat{1}) = X$ .

□

**Proposition 25.** The Euler characteristic  $\chi|CX|$  of a finite poset  $(X, \leq)$  is given by

$$\chi|CX| = \sum_{x, y \in X} \mu[x, y].$$

*Proof.* The result follows from the identity

$$|C_0 X| = |X| = \sum_{x \in X} \mu[x, x],$$

and the fact that for  $n \geq 1$  we have the identity

$$C_n X = \bigsqcup_{x < y} C_{n-2}(x, y).$$

Thus

$$\begin{aligned} \chi|CX| &= \sum_{n \geq 0} (-1)^n |C_n X| = |C_0 X| + \sum_{n \geq 1} (-1)^n |C_n X| = \\ &= \sum_{x \in X} \mu[x, x] + \sum_{n \geq 1} \left( \sum_{x < y} (-1)^n |C_{n-2}(x, y)| \right) = \\ &= \sum_{x \in X} \mu[x, x] + \sum_{x < y} \left( \sum_{n \geq 1} (-1)^{n-2} |C_{n-2}(x, y)| \right) = \\ &= \sum_{x \in X} \mu[x, x] + \sum_{x < y} \mu[x, y] = \sum_{x, y \in X} \mu[x, y]. \end{aligned}$$

□

For  $a \in X$ , we set  $X_{\geq a} = \{x \in X \mid x \geq a\}$ .

There is a structure of right  $[\mathbb{I}_X, R]$ -module on  $[X_{\geq a}, R]$  via the  $R$ -bilinear map

$$\star : [X_{\geq a}, R] \times [\mathbb{I}_X, R] \longrightarrow [X_{\geq a}, R]$$

sending a pair  $(f, g) \in [X_{\geq a}, R] \times [\mathbb{I}_X, R]$  to the map  $f \star g \in [X_{\geq a}, R]$  given by

$$f \star g(y) = \sum_{a \leq x \leq y} f(x)g[x, y].$$

**Theorem 26. (Möbius Inversion for Locally Finite Posets)**

Given  $a \in X$  and  $f, g \in [X_{\geq a}, R]$ , we have that  $f = g \star \xi$  if and only if  $g = f \star \mu$ , or equivalently

$$f(y) = \sum_{a \leq x \leq y} g(x) \quad \text{if and only if} \quad g(y) = \sum_{a \leq x \leq y} f(x)\mu[x, y].$$

**Remark 27.** If  $X$  is a finite set, then  $[X, R]$  is a right  $[\mathbb{I}_X, R]$ -module and we have for  $f, g \in [X, R]$  that:

$$f(y) = \sum_{x \leq y} g(x) \quad \text{if and only if} \quad g(y) = \sum_{x \leq y} f(x)\mu[x, y].$$

Fix a finite poset  $(X, \leq)$ . The set of maps  $[X \times X, R]$  is naturally an  $R$ -algebra (square matrices indexed by  $X$ ) with the product

$$fg(x, z) = \sum_{y \in X} f(x, y)g(y, z).$$

There is a natural embedding  $[\mathbb{I}_X, R] \longrightarrow [X \times X, R]$  of algebras sending  $f \in [\mathbb{I}_X, R]$  to the map  $f : X \times X \longrightarrow R$  given by

$$f(x, y) = \begin{cases} f[x, y] & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we may regard  $[\mathbb{I}_X, R]$  as a subalgebra of  $[X \times X, R]$ .

A map  $f \in [X \times X, R]$  is called transitive if for  $n \geq 0$ , and  $x_0, \dots, x_n \in X$  we have:

$$f(x_0, x_n) = 0 \quad \text{implies that} \quad f(x_0, x_1)f(x_1, x_2)\dots f(x_{n-1}, x_n) = 0.$$

Leinster has shown in [22, 23] the following results.

**Lemma 28.** Let  $f \in [X \times X, R]$  be an invertible and transitive map, then

$$f(x, y) = 0 \quad \text{implies that} \quad f^{-1}(x, y) = 0.$$

**Theorem 29.** Let  $f \in [\mathbb{I}_X, R] \subseteq [X \times X, R]$  be a transitive map, then  $f$  is invertible in  $[\mathbb{I}_X, R]$  if and only if  $f$  is invertible in  $[X \times X, R]$ .

So far we have regarded  $\mathbb{I}_X$  as a set, yet  $\mathbb{I}_X$  may be naturally regarded as a full subcategory of the category of finite posets and increasing maps. Thus we let  $\underline{\mathbb{I}}_X$  be the set of isomorphism classes of intervals in  $X$ . The coproduct and counit on  $< \mathbb{I}_X >$  descent to a coproduct and counit on  $< \underline{\mathbb{I}}_X >$ , giving an algebra structure  $\star$  to

$$< \underline{\mathbb{I}}_X >^* = [\underline{\mathbb{I}}_X, R],$$

the set of maps from  $\underline{\mathbb{I}}_X$  to  $R$ , or equivalently, the set of maps from  $\mathbb{I}_X$  to  $R$  invariant under isomorphisms. We call  $([\underline{\mathbb{I}}_X, R], \star)$  the reduced incidence algebra of  $(X, \leq)$ .

**Example 30.** Let  $(\mathbb{N}_+, |)$  be the poset of positive natural numbers with the order

$$n|m \quad \text{if and only if} \quad n \text{ divides } m.$$

The reduced incidence algebra  $([\mathbb{I}, R], \star)$  is isomorphic to the Dirichlet algebra  $\mathbb{D}_R$  via the map sending  $f \in [\mathbb{I}, R]$  to

$$D_f = \sum_{n \in \mathbb{N}_+} \frac{f[1, n]}{n^s}.$$

Indeed, since  $[d, n] \simeq [1, n/d]$  whenever  $d|n$ , we have for  $f, g \in [\mathbb{I}, R]$  that

$$\begin{aligned} D_{f \star g} &= \sum_{n \in \mathbb{N}_+} \frac{f \star g[\overline{1, n}]}{n^s} = \sum_{n \in \mathbb{N}_+} \left( \sum_{d|n} f[\overline{1, d}] g[\overline{d, n}] \right) \frac{1}{n^s} = \\ &= \sum_{n \in \mathbb{N}_+} \left( \sum_{d|n} f[\overline{1, d}] g[\overline{1, n/d}] \right) \frac{1}{n^s} = D_f D_g. \end{aligned}$$

**Example 31.** The reduced incidence algebra of the poset  $(\mathbb{N}, \leq)$  is isomorphic to  $R[[x]]$  via the map sending  $f \in [\mathbb{I}, R]$  to

$$\widehat{f} = \sum_{n=0}^{\infty} f[\overline{0, n}] x^n.$$

Indeed, since  $[k, n] \simeq [0, n-k]$  for  $k \leq n$ , we have for  $f, g \in [\mathbb{I}, R]$  that:

$$\widehat{f \star g} = \sum_{n=0}^{\infty} (f \star g)[\overline{0, n}] x^n = \sum_{n=0}^{\infty} \left( \sum_{k \leq n} f[\overline{0, k}] g[\overline{0, n-k}] \right) x^n = \widehat{f} \widehat{g}.$$

**Example 32.** The reduced incidence algebra of the poset  $(P_f \mathbb{N}, \subseteq)$  of finite subsets of  $\mathbb{N}$  ordered by inclusion is isomorphic to the divided powers algebra

$$R \langle\langle x_0, x_1, \dots, \frac{x_n}{n!}, \dots \rangle\rangle,$$

with product given by

$$\frac{x_n}{n!} \frac{x_m}{m!} = \binom{n+m}{n} \frac{x^{n+m}}{(n+m)!},$$

via the map sending  $f \in [\mathbb{I}, R]$  to

$$\widehat{f} = \sum_{n=0}^{\infty} f[\overline{\emptyset, [n]}] \frac{x^n}{n!}.$$

Indeed, since  $[a, b] \simeq [\emptyset, b \setminus a]$ , we have for  $f, g \in [\mathbb{I}, R]$  that:

$$\begin{aligned} \widehat{f \star g} &= \sum_{n=0}^{\infty} (f \star g)[\overline{\emptyset, [n]}] x^n = \sum_{n=0}^{\infty} \left( \sum_{a \subseteq [n]} f[\overline{\emptyset, a}] g[\overline{\emptyset, [n] \setminus a}] \right) \frac{x^n}{n!} = \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \binom{n}{k} f[\overline{\emptyset, [k]}] g[\overline{\emptyset, [n-k]}] \right) \frac{x^n}{n!} = \widehat{f} \widehat{g}. \end{aligned}$$

Next we show that the Möbius function admits a Hopf theoretical interpretation. This connection has been developed by Shmitt [31] in a remarkable series of papers. For more on Hopf algebras the reader may consult [8, 17].

**Theorem 33.** For a locally finite poset  $(X, \leq)$  we let  $R[[x_{[a,b]}]]$  be the  $R$ -algebra of formal power series in the variables  $x_{[a,b]}$  with  $a < b$  in  $X$ , i.e. one variable for each interval in  $X$  with different endpoints. The structural maps given below turn  $R[[x_{[a,b]}]]$  into a Hopf algebra such that for  $a < b$  in  $X$  we have

$$\mu_X[a, b] = Sx_{[a,b]}(1).$$

*Proof.* The counit, coproduct, and antipode on  $R[[x_{[a,b]}]]$  are given, respectively, on generators by

$$\begin{aligned} \epsilon 1 &= 1 \quad \text{and} \quad \epsilon x_{[a,b]} = 0, \\ \Delta 1 &= 1 \otimes 1 \quad \text{and} \quad \Delta x_{[a,b]} = 1 \otimes x_{[a,b]} + \sum_{a < c < b} x_{[a,c]} \otimes x_{[c,b]} + x_{[a,b]} \otimes 1, \\ S1 &= 1 \quad \text{and} \quad Sx_{[a,b]} = \sum_{n \geq 1} \sum_{\{a=a_0 < a_1 < \dots < a_n=b\}} (-1)^n x_{[a_0,a_1]} \dots x_{[a_{n-1},a_n]}. \end{aligned}$$

□

**Corollary 34.** For a locally finite poset  $(X, \leq)$  we let  $R[[x_{\overline{[a,b]} }]]$  be the  $R$ -algebra of formal power series in the variables  $x_{\overline{[a,b]}}$  with  $\overline{[a,b]} \in \mathbb{I}$  and  $a < b$ , i.e. one variable for each isomorphism class of intervals in  $X$  with different endpoints. The structural maps from Theorem 33 induce structural maps on  $R[[x_{\overline{[a,b]} }]]$  turning it into a Hopf algebra such that for  $a < b$  in  $X$  we have:

$$\mu_X[\overline{[a,b]}] = Sx_{\overline{[a,b]}}(1).$$

Next we state and prove a discrete analogue of the Gauss-Bonnet theorem for finite posets, and discuss how this construction relates to the Möbius function. Similar results for graphs have been developed by Knill [18]. We recall that the Gauss-Bonnet theorem describes the Euler characteristic of a compact smooth manifold  $M$  as the integral of a top differential form on  $M$ .

We have already defined the space of  $\mathbb{Z}$ -linear  $n$ -chains on a finite poset  $(X, \leq)$  as the free  $\mathbb{Z}$ -module  $\langle C_n X \rangle$  generated by the set  $C_n X$  of linearly ordered subsets of  $X$  of length  $n+1$ . The discrete analogue  $\Omega^n X$  for the differential forms of degree  $n$  is simply the dual  $\mathbb{Z}$ -module  $\langle C_n X \rangle^*$ , or equivalently, the  $\mathbb{Z}$ -module  $[C_n X, \mathbb{Z}]$ . We denote by

$$\int_c \omega$$

the natural pairing between  $c \in \langle C_n X \rangle$  and  $\omega \in \Omega^n X = \langle C_n X \rangle^*$ . We use the same notation for the trivially extended pairing

$$\int : \langle C_{\geq 0} X \rangle \times \Omega X \longrightarrow \mathbb{Z}, \quad \text{where}$$

$$\langle CX \rangle = \bigoplus_{n \geq 0} \langle C_n X \rangle \quad \text{and} \quad \Omega X = \bigoplus_{n \geq 0} \Omega^n X.$$

Let  $\mathbb{M}$  be the set of all maximal linearly ordered subsets of  $X$ . The fundamental class of  $X$  is the  $\mathbb{Z}$ -linear chain  $[X] \in \langle CX \rangle$  given by:

$$[X] = \sum_{m \in \mathbb{M}} m.$$

For  $c \in CX$ , we set  $\mathbb{M}_c = \{m \in \mathbb{M} \mid c \subseteq m\}$ . The Euler class  $e_X \in \Omega X$  of  $X$  is such that  $e_X(c) = 0$  for  $c \notin \mathbb{M}$ , and for  $m \in \mathbb{M}$  we have that

$$e_X(m) = \sum_{\emptyset \neq c \subseteq m} \frac{(-1)^{|c|+1}}{|\mathbb{M}_c|}.$$

Similarly, the reduced Euler class  $\tilde{e}_X \in \Omega X$  vanishes on non-maximal linearly ordered subsets, and is given on a maximal chain  $m \in \mathbb{M}$  by

$$\tilde{e}_X(m) = \sum_{c \subseteq m} \frac{(-1)^{|c|+1}}{|\mathbb{M}_c|}.$$

**Theorem 35. (Gauss-Bonnet for Finite Posets)**

Let  $(X, \leq)$  be a finite poset and  $|CX|$  be its geometric realization. We have that

$$\chi|CX| = \int_{[X]} e_X, \quad \text{and} \quad \tilde{\chi}|CX| = \int_{[X]} \tilde{e}_X.$$

*Proof.* We show the latter identity, the proof of the former being analogous:

$$\begin{aligned} \tilde{\chi}|CX| &= \sum_{n \geq -1} (-1)^n |C_n X| = \sum_{c \in CX} (-1)^{|c|+1} = \sum_{c \in CX} \sum_{m \in \mathbb{M}_c} \frac{(-1)^{|c|+1}}{|\mathbb{M}_c|} = \\ &= \sum_{m \in \mathbb{M}} \sum_{c \subseteq m} \frac{(-1)^{|c|+1}}{|\mathbb{M}_c|} = \sum_{m \in \mathbb{M}} \tilde{e}_X(m) = \int_{[X]} \tilde{e}_X. \end{aligned}$$

□

**Corollary 36.** Let  $(X, \leq)$  be a poset and  $x < y$  in  $X$ , then

$$\mu[x, y] = \tilde{\chi}|C(x, y)| = \int_{[(x, y)]} \tilde{e}_{(x, y)},$$

where  $(x, y)$  is the poset  $(x, y) = \{z \mid x < z < y\} \subseteq X$  with the induced order.

## 4 Locally Finite Reflexive Directed Graphs

We proceed to consider our second example of a Möbius theory. Recall that a directed graph is a pair  $(X, E)$  of sets together with a (source, target) map

$$(s, t) : E \longrightarrow X \times X.$$

For  $x, y \in X$  we set

$$E(x, y) = \{e \in E \mid (s, t)(e) = (x, y)\}.$$

A directed graph  $(X, E)$  is called reflexive if  $E(x, x) \neq \emptyset$  for all  $x \in X$ .

A walk  $\gamma$  of length  $l(\gamma) = n \in \mathbb{N}_+$  in  $(X, E)$  is a sequence  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n) \in E^n$  such that  $t(\gamma_i) = s(\gamma_{i+1})$  for  $i \in [n-1]$ , and  $s(\gamma_i) \neq t(\gamma_i)$  for  $i \in [n]$ . We say that the walk  $\gamma$  begins at  $s(\gamma_1)$  and ends at  $t(\gamma_n)$ .

For  $x, y \in X$ , we let  $W(x, y)$  ( $W_k(x, y)$ ) be the set of walks (of length  $k$ ) from  $x$  to  $y$ .

A circuit  $\gamma$  of length  $n \in \mathbb{N}_{\geq 2}$  in  $(X, E)$  is a walk of length  $n$  such that  $t(\gamma_n) = s(\gamma_1)$  and  $t(\gamma_i) \neq t(\gamma_j)$  for  $i \neq j$ .

**Definition 37.** A graph  $(X, E)$  is locally finite if  $E(x, x)$  and  $W(x, y)$  are finite sets for all  $x, y \in X$ .

Note that a locally finite directed graph has no circuits. Moreover, a finite directed graph is locally finite if and only if it has not circuits.

**Definition 38.** Given a locally finite reflexive directed graph  $(X, E)$  we let  $\leq$  be the relation on  $X$  such that  $x \leq y$  if and only if  $x = y$  or there is walk in  $(X, E)$  from  $x$  to  $y$ .

**Proposition 39.** For a locally finite reflexive directed graph  $(X, E)$  the pair  $(X, \leq)$ , with the relation  $\leq$  on  $X$  from Definition 38, is a locally finite poset.

*Proof.* Reflexivity is immediate. Transitivity follows since the concatenation of walks is a walk. Anti-symmetry is a consequence of the fact that  $(X, E)$ , a locally finite graph, has no circuits. Thus  $(X, \leq)$  is a poset. It is locally finite since we have an injective map that associates to each  $z$  with  $x < z < y$  the walk  $x \longrightarrow z \longrightarrow y$  from  $x$  to  $y$ . Since  $W(x, y)$  is a finite set, then necessarily the interval  $[x, y]$  is a finite set as well.

□



Thus for any locally finite reflexive directed graph  $(X, E)$  we have the incidence algebra  $[\mathbb{I}_{(X, \leq)}, R]$ . The incidence or adjacency map  $\xi \in [X \times X, R]$  of  $(X, E)$  is given by

$$\xi(x, y) = |E(x, y)|.$$

Clearly, we may regard  $\xi$  as an element of the incidence algebra  $[\mathbb{I}_{(X, \leq)}, R]$ . Note that the adjacency map  $\xi_{(X, E)}$  of the graph  $(X, E)$  in general is not equal to the adjacency map  $\xi_{(X, \leq)}$  of the associated poset  $(X, \leq)$ .

**Theorem 40. (Möbius Function for Locally Finite Reflexive Directed Graphs)**

Let  $(X, E)$  be a locally finite reflexive directed graph. The adjacency map  $\xi$  of  $(X, E)$  is invertible in  $[\mathbb{I}_{(X, \leq)}, R]$ ; its inverse  $\mu$ , called the Möbius function of  $(X, E)$ , is such that

$$\mu[x, x] = \frac{1}{|E(x, x)|}$$

and for  $x \neq y$  in  $X$  we have:

$$\mu[x, y] = \sum_{\gamma \in W(x, y)} \frac{(-1)^{l(\gamma)}}{|E(s(\gamma_1), s(\gamma_1))| |E(t(\gamma_1), t(\gamma_1))| \dots |E(t(\gamma_{l(n)}), t(\gamma_{l(n)}))|}.$$

*Proof.* Follows from Theorem 15 the considerations above.  $\square$

Recall that given  $a \in X$  there is a structure of right  $[\mathbb{I}_{(X, \leq)}, R]$ -module on  $[X_{\geq a}, R]$  via the  $R$ -bilinear map  $\star : [X_{\geq a}, R] \times [\mathbb{I}_X, R] \longrightarrow [X_{\geq a}, R]$  given by

$$f \star g(y) = \sum_{a \leq x \leq y} f(x)g[x, y].$$

**Corollary 41. (Möbius Inversion for Locally Finite Reflexive Directed Graphs)**

Fix  $a \in X$ . For  $f, g \in [X_{\geq a}, R]$  we have that

$$g(y) = \sum_{e \in E, a \leq se, te=y} f(se) \quad \text{if and only if} \quad f(y) = \sum_{a \leq x \leq y} g(x)\mu(x, y).$$

*Proof.* We have that  $g = f \star \xi$  if and only if  $g = g \star \mu$ . The result follows since

$$g(y) = f \star \xi(y) = \sum_{a \leq x \leq y} f(x)\xi[x, y] = \sum_{e \in E, a \leq se, te=y} f(se).$$

$\square$

Given directed graphs  $(X, E)$  and  $(Y, F)$ , the product graph  $(X \times Y, E \times F)$  is such that an edge from  $(a_1, b_1)$  to  $(a_2, b_2)$  in  $X \times Y$  is the same as a pair of edges  $(e, f) \in E \times F$  where  $e$  is an edge from  $a_1$  to  $a_2$  in  $X$ , and  $f$  is an edge from  $b_1$  to  $b_2$  in  $Y$ .

**Proposition 42.** Let  $(X, E)$  and  $(Y, F)$  be finite reflexive circuit-less directed graphs, the the product graph  $(X \times Y, E \times F)$  is also a finite reflexive circuit-less, and we have an isomorphism of posets

$$(X \times Y, \leq) \simeq (X, \leq) \times (Y, \leq)$$

thus we have a natural isomorphism of algebras

$$([\mathbb{I}_{(X \times Y, \leq)}, R], \star) \simeq ([\mathbb{I}_{(X, \leq)}, R], \star) \otimes ([\mathbb{I}_{(Y, \leq)}, R], \star).$$

Moreover, we have that

$$\xi_{X \times Y}[(a_1, b_1), (a_2, b_2)] = \xi_X[a_1, a_2] \xi_Y[b_1, b_2]$$

and thus

$$\mu_{X \times Y}[(a_1, b_1), (a_2, b_2)] = \mu_X[a_1, a_1] \mu_Y[b_1, b_2].$$

Let  $(X, \leq)$  be a locally finite poset and consider the cover relation  $\prec$  on  $X$  given by

$$x \prec y \text{ if and only if } x < y \text{ and there is no } z \in X \text{ such that } x < z < y.$$

Consider the map  $\eta \in [\mathbb{I}_{(X, \leq)}, R]$  given by

$$\eta[x, y] = \begin{cases} 1 & \text{if } x = y, \\ -1 & \text{otherwise.} \end{cases}$$

It follows from Theorem 40 that  $\eta \in [\mathbb{I}_{(X, \leq)}, R]$  is invertible and its inverse  $\eta^{-1} \in [\mathbb{I}_{(X, \leq)}, R]$  is such that

$$\eta^{-1}[x, x] = 1 \quad \text{and} \quad \eta^{-1}[x, y] = |\mathbb{M}[x, y]|,$$

where  $\mathbb{M}[x, y]$  is the set of maximal linearly ordered subsets of the interval  $[x, y] \subseteq X$ .

Fix  $a \in X$ , the finite difference operator

$$\Delta : [X_{\geq a}, R] \longrightarrow [X_{\geq a}, R]$$

is given for  $f \in [X, R]$  and  $y \in X_{\geq a}$  by

$$\Delta f(y) = f(y) - \sum_{a \leq x \prec y} f(x) = f \star \eta(y).$$

The Möbius inversion formula tell us that

$$\Delta f = g \quad \text{if and only if} \quad f(y) = \sum_{a \leq x \leq y} |\mathbb{M}[x, y]| g(x).$$

## 5 Locally Finite Categories

In this section we introduce the coarse Möbius theory for categories (although using a different terminology and focusing on a concrete case) developed by Leinster [22, 23]. This sort of Möbius theory depends only on the underlying graph of the category.

Given a category  $C$  we let  $C_0$  be the collection of its objects. Abusing notation we usually write  $x \in C$  instead of  $x \in C_0$ . Let  $C_1$  be the collection of morphisms in  $C$ , and  $C(x, y)$  be the set of morphisms in  $C$  from  $x$  to  $y$ . The source and target maps

$$(s, t) : C_1 \longrightarrow C_0 \times C_0$$

together with the map  $1 : C_0 \longrightarrow C_1$ , sending each object  $x \in C$  to its identity  $1_x \in C(x, x)$ , give  $C$  the structure of a reflexive directed graph. Of course, in a category we have in addition the composition maps

$$\circ : C(x, y) \times C(y, z) \longrightarrow C(x, z)$$

sending a pair of morphisms  $(f, g) \in C(x, y) \times C(y, z)$  to its composition  $gf$ . Composition of morphisms is associative and unital in the sense that

$$(hg)f = h(gf) \quad \text{and} \quad 1_y f = f = f 1_x, \quad \text{for} \quad f, g, h \in C(x, y) \times C(y, z) \times C(y, z).$$

The notation  $x \xrightarrow{f} y$  means that  $f$  is a morphism in  $C(x, y)$ , and  $x = sf$  and  $y = tf$ . For  $x, y \in C$  we set

$$[x, y] = \{z \in C_0 \mid \text{there is a diagram } x \longrightarrow z \longrightarrow y \text{ in } C\}.$$

**Definition 43.** A category  $C$  is locally finite if the following conditions hold:

- $C(x, y)$  is a finite set for all  $x, y \in C$ .
- If there is a diagram  $x \longrightarrow y \longrightarrow x$  in  $C$ , then  $x = y$ .
- For  $x, y \in C$ , the set  $[x, y]$  is a finite.

**Remark 44.** Locally finite categories are skeletal.

**Example 45.** A finite category (i.e. a category with a finite set of morphisms) is locally finite if and only if in any diagram  $x \longrightarrow y \longrightarrow x$ , we have that  $x = y$ .

**Example 46.** Let  $C$  be a category with  $C(x, y)$  finite for all  $x, y \in C$ , let  $(X, \leq)$  be a locally finite poset, and let  $F : X \longrightarrow C$  be a functor. The category  $X_F$  with objects  $X$  and morphisms given by

$$X_F(x, y) = \begin{cases} C(F(x), F(y)) & \text{if } x \leq y, \\ \emptyset & \text{otherwise,} \end{cases}$$

is locally finite.

**Proposition 47.** A category is locally finite if and only if its graph is locally finite.

*Proof.* Let  $C$  be a locally finite category. Then  $C(x, x)$  is a finite set for all  $x \in C$ . For  $x \neq y \in C$ , consider the set  $W(x, y)$  of walks from  $x$  to  $y$ . The objects in a walk

$$x \longrightarrow x_1 \longrightarrow \dots \longrightarrow x_n \longrightarrow y$$

are all distinct because in any configuration of the form

$$x \longrightarrow x_1 \longrightarrow \dots \longrightarrow x_k \longrightarrow x$$

we must have that  $x = x_1 = \dots = x_k$ . Notice that if an object  $z \in C$  appears in a walk from  $x$  to  $y$ , then  $z \in [x, y]$ . Since  $[x, y]$  is a finite set, then necessarily  $W(x, y)$  is also a finite set. In particular we have that  $W(x, x) = \emptyset$ . Thus the graph of  $C$  is locally finite.

Conversely, assume that the graph of  $C$  is locally finite. The sets  $C(x, x)$  are finite by definition. For  $x \neq y$ , the sets  $C(x, y)$  and  $[x, y]$  are finite, since there is an injective map  $C(x, y) \longrightarrow W(x, y)$ , and a surjective map

$$\{x\} \sqcup W_2(x, y) \sqcup \{y\} \longrightarrow [x, y].$$

Recall that a locally finite graph has no circuits, thus in any configuration  $x \longrightarrow y \longrightarrow x$  we must have that  $x = y$ . In particular  $[x, x] = \{x\}$ . □

**Lemma 48.** Let  $C$  be a category with  $C(x, y)$  finite for  $x, y \in C$ . Let  $\leq$  be the relation on  $C_0$  given by  $x \leq y$  if and only if there is a morphism  $x \longrightarrow y$  in  $C$ . Then  $C$  is a locally finite category if and only if  $x \leq y$  defines a locally finite partial order on  $C_0$ .

*Proof.* Assume  $C$  is a locally finite category, then  $C_0$  is a poset with the ordering from statement of the theorem. Reflexivity and transitivity are immediate. Anti-symmetry follows from the second property in Definition 43. The poset  $(C_0, \leq)$  is locally finite by the third property in Definition 43.

Conversely, if  $(C_0, \leq)$  is a locally finite poset, then a diagram  $x \longrightarrow y \longrightarrow x$  implies that  $x = y$  by the reflexivity of  $\leq$ . The interval  $[x, y]$  is finite since

$$[x, y] = \{z \in C_0 \mid \text{there is a diagram } x \longrightarrow z \longrightarrow y \text{ in } C\} = \{z \in C_0 \mid x \leq z \leq y\}.$$

□

Thus for any locally finite category  $C$  we have the partially order set  $(C_0, \leq)$  and the corresponding incidence algebra  $(\mathbb{I}_{(C_0, \leq)}, R, \star)$ . The incidence or adjacency map of  $C$

$$\xi : C_0 \times C_0 \longrightarrow R$$

is given by

$$\xi[x, y] = |C(x, y)|$$

and lies naturally in  $[\mathbb{I}_{(C_0, \leq)}, R]$ .

**Theorem 49. (Möbius Function for Locally Finite Categories)**

Let  $C$  be a locally finite category. The adjacency map  $\xi$  of  $C$  is invertible in  $[\mathbb{I}_{(C_0, \leq)}, R]$ ; its inverse  $\mu$ , called the Möbius function of  $C$ , is such that

$$\mu[x, x] = \frac{1}{|C(x, x)|}$$

and for  $x \neq y \in C$  we have that

$$\begin{aligned} \mu[x, y] &= \sum_{n \geq 1} \sum_{x=x_0 < x_1 < \dots < x_n=y} \frac{(-1)^n |C(x_0, x_1)| \dots |C(x_{n-1}, x_n)|}{|C(x_0, x_0)| |C(x_1, x_1)| \dots |C(x_{n-1}, x_{n-1})| |C(x_n, x_n)|} = \\ &= \sum_{n \geq 1} \sum_{x=x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n=y} \frac{(-1)^n}{|C(x_0, x_0)| |C(x_1, x_1)| \dots |C(x_{n-1}, x_{n-1})| |C(x_n, x_n)|}, \end{aligned}$$

where in the second identity the sum ranges over all diagrams in  $C$  of the form

$$x = x_0 \longrightarrow x_1 \longrightarrow \dots \longrightarrow x_n = y.$$

*Proof.* The result follows from Theorem 15. □

Fix  $a \in C$ . Then  $[C_{0, \geq a}, R]$  is a right  $[\mathbb{I}_{(C_0, \leq)}, R]$ -module, thus we get the following result.

**Corollary 50. (Möbius Inversion for Locally Finite Categories)**

Fix  $a \in X$ . For  $f, g \in [C_{0, \geq a}, R]$  we have that

$$g(y) = \sum_{a \leq x \leq y} f(x) |C(x, y)| \quad \text{if and only if} \quad f(y) = \sum_{a \leq x \leq y} g(x) \mu(x, y).$$

Recall that if  $C$  and  $D$ , then objects in the product category  $C \times D$  are pairs  $(a, b)$  with  $a \in C$  and  $b \in D$ . Morphisms in  $C \times D$  are given by

$$C \times D((a_1, b_1), (a_2, b_2)) = C(a_1, a_2) \times D(b_1, b_2).$$

**Proposition 51.** Let  $C$  and  $D$  be locally finite categories, then the product  $C \times D$  is also a locally finite category, and we have a natural isomorphism of posets

$$\mathbb{I}_{((C \times D)_0, \leq)} \simeq \mathbb{I}_{(C_0, \leq)} \times \mathbb{I}_{(D_0, \leq)},$$

and thus a natural isomorphism of algebras

$$([\mathbb{I}_{((C \times D)_0, \leq)}, R], \star) \simeq ([\mathbb{I}_{(C_0, \leq)}, R], \star) \otimes ([\mathbb{I}_{(D_0, \leq)}, R], \star).$$

Moreover, we have that

$$\xi_{C \times D}[(a_1, b_1), (a_2, b_2)] = \xi_C[a_1, a_2] \xi_D[b_1, b_2]$$

and thus

$$\mu_{C \times D}[(a_1, b_1), (a_2, b_2)] = \mu_C[a_1, a_2] \mu_D[b_1, b_2].$$

**Example 52.** Let  $C$  be a locally finite category,  $a \in C$ , and  $F : C \rightarrow \text{set}$  be a functor from  $C$  to the category of finite sets. Consider the functor  $G : C \rightarrow \text{set}$  given on objects by

$$G(y) = \bigsqcup_{a \leq x \leq y} F(x) \times C(x, y).$$

By the Möbius inversion formula we have that

$$|F(y)| = \sum_{a \leq x \leq y} |G(x)| \mu(x, y).$$

## 6 Essentially Locally Finite Categories

The Möbius theory for locally finite categories developed in Section 5 although functorial under isomorphisms of categories, fails to be functorial under equivalences of categories. We recall that categories  $C$  and  $D$  are equivalent if there is a functor  $F : C \rightarrow D$  that is essentially surjective, full and faithful [26]. A category may fail to be locally finite and yet be equivalent to a locally finite category. For example, the category with two objects, a unique isomorphism between them, and identities as the only endomorphism, is equivalent to the category with one object and one morphism. The latter is locally finite whereas the former is not.

Given a category  $C$  and objects  $x, y \in C$ , the notation  $x \simeq y$  means that  $x$  and  $y$  are isomorphic objects. Let  $\overline{C}$  be set of isomorphism classes of objects in  $C$ , i.e.  $\overline{C}$  is the quotient set  $C_0 / \simeq$ . A typical element of  $\overline{C}$  is denoted by  $\overline{x}$ , meaning that we have an equivalence class  $\overline{x} \in \overline{C}$  and that we have chosen a representative object  $x \in \overline{x}$ . For  $\overline{x}, \overline{y} \in \overline{C}$  we set

$$[\overline{x}, \overline{y}] = \{ \overline{z} \in \overline{C} \mid \text{there is a diagram } x \rightarrow z \rightarrow y \text{ in } C \}.$$

**Definition 53.** A category  $C$  is essentially locally finite if the following conditions hold:

- $C(x, y)$  is a finite set for  $x, y \in C$ .
- If we have a diagram  $x \rightarrow y \rightarrow x$  in  $C$ , then  $x \simeq y$ .
- For  $x, y \in C$ , the set  $[\overline{x}, \overline{y}]$  is finite.

**Remark 54.** In the applications we often find a stronger version of the second property in Definition 53: the arrows in any configuration  $x \longrightarrow y \longrightarrow x$  are isomorphisms. We call categories with such a property isocyclic, i.e. all cycles of morphisms in  $C$  are formed by isomorphisms. Not all essentially locally finite categories are isocyclic, e.g. a finite monoid regarded as a category.

**Lemma 55.** Let  $C$  be a category with  $C(x, y)$  finite for  $x, y \in C$ . Let  $\leq$  be the relation on  $\overline{C}$  given by  $\overline{x} \leq \overline{y}$  if and only if there is a morphism  $x \longrightarrow y$  in  $C$ . Then  $C$  is an essentially locally finite category if and only if  $\leq$  is a locally finite partial order on  $\overline{C}$ .

*Proof.* Reflexivity and transitivity of  $\leq$  are obvious. Antisymmetry is equivalent to the second property of Definition 53. The third property in Definition 53 is equivalent to local finiteness.  $\square$

**Proposition 56.** A category  $C$  is essentially locally finite if and only if  $C$  is equivalent to a locally finite category.

*Proof.* Assume  $C$  is an essentially locally finite category. Let  $S$  be a full subcategory of  $C$  whose objects include one and only one representative of each isomorphism class of  $C$ . The category  $S$  is equivalent to  $C$  to and skeletal, thus it is a locally finite category.

Conversely, let  $S$  be a locally finite category and  $F : S \longrightarrow C$  an equivalence of categories. For objects  $y_1, y_2 \in C$  there are objects  $x_1, x_2 \in S$  such that

$$F(x_1) \simeq y_1 \quad \text{and} \quad F(x_2) \simeq y_2.$$

Moreover, we have bijective maps

$$C(x_1, x_2) \longrightarrow C(y_1, y_2) \quad \text{and} \quad [x_1, x_2] \longrightarrow [y_1, y_2].$$

Thus  $C(y_1, y_2)$  and  $[y_1, y_2]$  are finite sets. If there is a diagram  $y_1 \longrightarrow y_2 \longrightarrow y_1$  in  $C$ , then there is a corresponding diagram  $x_1 \longrightarrow x_2 \longrightarrow x_1$  in  $S$ , thus  $x_1 = x_2$  and  $y_1 \simeq y_2$ .  $\square$

**Example 57.** Let  $C$  be a subcategory of the category of finite sets and maps. Let  $IC$  be the subcategory of  $C$  with the same objects as  $C$  and such that  $f \in IC(x, y) \subseteq C(x, y)$  if and only if  $f$  is an injective map from  $x$  to  $y$ . Dually, let  $SC$  be the subcategory of  $C$  with the same objects as  $C$  and such that  $f \in SC(x, y) \subseteq C(x, y)$  if and only if  $f$  is a surjective map. The categories  $IC$  and  $SC$  are isocyclic and essentially locally finite categories.

Let  $\mathbb{I}$  be the category of finite sets and injective maps. A combinatorial presheaf  $P$  is contravariant functor  $P : \mathbb{I} \longrightarrow \text{set}$ . The Grothendieck category of elements  $\mathbb{I}_P$  has for objects pairs  $(x, a)$  where  $a \in Px$ . A morphism

$$(x, a) \longrightarrow (y, b) \in \mathbb{I}_P$$

is an injective map  $f : x \longrightarrow y$  such that

$$b|_x = P_f b = a.$$

**Proposition 58.** For any combinatorial presheaf  $P$  the category of elements  $\mathbb{I}_P$  is isocyclic and essentially locally finite.

*Proof.* A composition of maps between finite sets is a bijection if and only if the maps are bijections. Whenever we have an element  $\overline{(z, c)}$  in an interval  $[(x, a), \overline{(y, b)}]$  of  $\mathbb{I}_P$  we may assume, using isomorphic representations, that

$$x \subseteq z \subseteq y, \quad a = c|_x c = b|_z \quad \text{and} \quad c = b|_z.$$

Thus there is only a finite number of choices for  $\overline{(z, c)}$  and  $\mathbb{I}_P$  is an essentially locally finite category.  $\square$

Let  $\mathbb{B}$  be the category of finite sets and bijections, and let  $O$  be an operad in the category of finite sets. Thus  $O$  is a functor

$$O : \mathbb{B}_+ \longrightarrow \text{set},$$

from the category  $\mathbb{B}_+$  of non-empty finite sets and bijections to the category of finite sets, together with a distinguished element  $1 \in O[1]$  and suitable composition maps

$$m_\pi : O(\pi) \times \prod_{b \in \pi} O(b) \longrightarrow O(x),$$

where  $\pi$  is a partition of the finite set  $x$ . The reader may consult [11, 25] and the references there in for details. Let  $\mathbb{S}_O$  be the category whose objects are finite sets, and such that a morphism  $(f, a) \in \mathbb{S}_O(x, y)$  is a surjective map  $f : x \longrightarrow y$  together with an element

$$a \in \prod_{j \in y} O(f^{-1}j).$$

Composition of morphisms is defined with the help of the operadic compositions as follows. Suppose we have morphisms

$$x \xrightarrow{(f, a)} y \xrightarrow{(g, b)} z \quad \text{where}$$

- $f : x \longrightarrow y$  is a surjective map, and  $a = (a_j)_{j \in y}$  with  $a_j \in O(f^{-1}j)$ ,
- $g : y \longrightarrow z$  is a surjective map, and  $b = (b_k)_{k \in z}$  with  $b_k \in O(g^{-1}k)$ .

The composition

$$(g, b) \circ (f, a) = (gf, ba)$$



is such that  $gf : x \longrightarrow z$  is the composition map, and for  $k \in z$  the element  $(ba)_k \in O((gf)^{-1}k)$  is defined as follows. Since

$$(gf)^{-1}k = \bigsqcup_{j \in g^{-1}k} f^{-1}j \quad \text{we get a map}$$

$$m : O(\{f^{-1}j\}_{j \in g^{-1}k}) \times \prod_{j \in g^{-1}k} O(f^{-1}j) \longrightarrow O((gf)^{-1}k),$$

or equivalently a map

$$m : O(g^{-1}k) \times \prod_{j \in g^{-1}k} O(f^{-1}j) \longrightarrow O((gf)^{-1}k).$$

We let  $(ba)_k \in O((gf)^{-1}k)$  be given by

$$(ba)_k = m(b_k, (a_j)_{j \in g^{-1}k}).$$

**Proposition 59.** For any operad  $O$  of finite sets the category  $\mathbb{S}_O$  is isocyclic and essentially locally finite.

*Proof.* Again a composition of maps between finite sets is a bijection if and only if the maps are bijections. Whenever we have an element  $\overline{(z, c)}$  in an interval  $[\overline{(x, a)}, \overline{(y, b)}]$  of  $\mathbb{S}_O$  we may assume that  $z$  is a partition of  $x$ , using isomorphic representations, thus there is only a finite number of choices for  $\overline{(z, c)}$  and  $\mathbb{S}_O$  is a locally finite category.  $\square$

For any essentially locally finite category  $C$  we have the partially order set  $(\overline{C}, \leq)$  and the incidence algebra  $([\mathbb{I}_{(\overline{C}, \leq)}, R], \star)$ . The incidence or adjacency map  $\xi : \overline{C} \times \overline{C} \longrightarrow R$  of  $C$

$$\xi(\overline{x}, \overline{y}) = |C(x, y)|$$

lives naturally in  $[\mathbb{I}_{(\overline{C}, \leq)}, R]$ .

**Theorem 60. (Möbius Function for Essentially Locally Finite Categories)**

Let  $C$  be an essentially locally finite category. The adjacency map  $\xi$  is invertible in  $[\mathbb{I}_{(\overline{C}, \leq)}, R]$ ; its inverse  $\mu$ , called the Möbius function of  $C$ , is such that

$$\mu[\overline{x}, \overline{x}] = \frac{1}{|C(x, x)|}$$

and for  $\overline{x} \neq \overline{y}$  in  $C$  we have that

$$\mu[\overline{x}, \overline{y}] = \sum_{n \geq 1} \sum_{\overline{x} = \overline{x_0} < \overline{x_1} < \dots < \overline{x_n} = \overline{y}} (-1)^n \frac{|C(x_0, x_1)| \dots |C(x_{n-1}, x_n)|}{|C(x_0, x_0)| |C(x_1, x_1)| \dots |C(x_{n-1}, x_{n-1})| |C(x_n, x_n)|}.$$

*Proof.* Follows from Theorem 15.  $\square$

For  $\bar{a} \in \bar{C}$ , we have that  $[\bar{C}_{\geq \bar{a}}, R]$  is a right  $[\mathbb{I}_{(\bar{C}, \leq)}, R]$ -module.

**Corollary 61. (Möbius Inversion for Essentially Locally Finite Categories)**

Fix  $\bar{a} \in \bar{C}$ . For  $f, g \in [\bar{C}_{\geq \bar{a}}, R]$  we have that

$$g(\bar{y}) = \sum_{\bar{a} \leq \bar{x} \leq \bar{y}} f(\bar{x}) |C(\bar{x}, \bar{y})| \quad \text{if and only if} \quad f(\bar{y}) = \sum_{\bar{a} \leq \bar{x} \leq \bar{y}} g(\bar{x}) \mu(\bar{x}, \bar{y}).$$

**Proposition 62.** Let  $C$  and  $D$  be finite essentially locally finite categories, then  $C \times D$  is a finite essentially locally finite category, and we have a natural isomorphism of posets

$$\mathbb{I}_{(\overline{C \times D}, \leq)} \simeq \mathbb{I}_{(\bar{C}, \leq)} \times \mathbb{I}_{(\bar{D}, \leq)}$$

thus we have a natural isomorphism of algebras

$$([\mathbb{I}_{(\overline{C \times D}, \leq)}, R], \star) \simeq ([\mathbb{I}_{(\bar{C}, \leq)}, R], \star) \otimes ([\mathbb{I}_{(\bar{D}, \leq)}, R], \star).$$

Moreover, we have that

$$\xi_{C \times D}[\overline{(a_1, b_1)}, \overline{(a_2, b_2)}] = \xi_C[\bar{a}_1, \bar{a}_2] \xi_D[\bar{b}_1, \bar{b}_2]$$

and thus

$$\mu_{C \times D}[\overline{(a_1, b_1)}, \overline{(a_2, b_2)}] = \mu_C[\bar{a}_1, \bar{a}_2] \mu_D[\bar{b}_1, \bar{b}_2].$$

The formula for the Möbius function  $\mu$  from Theorem 60 admits a nice conceptual understanding in the case of isocyclic categories which we proceed to formulate. We introduce first a few required mathematical notions.

The cardinality of finite sets can be viewed as an invariant under isomorphisms map

$$| \cdot | : \text{set} \longrightarrow \mathbb{N}$$

from the category of finite sets to the set of natural numbers, which satisfies

$$|\emptyset| = 0, \quad |[1]| = 1, \quad |x \sqcup y| = |x| + |y|, \quad \text{and} \quad |x \times y| = |x| |y|.$$

The notion of cardinality for finite sets admits a suitable extension [3, 5, 6] to the category  $\text{gpd}$  of essentially finite groupoids (i.e. groupoids equivalent to finite groupoids) via the invariant under equivalences map

$$| \cdot |_g : \text{gpd} \longrightarrow \mathbb{Q},$$

given by

$$|G|_g = \sum_{x \in \bar{G}} \frac{1}{|G(x, x)|}.$$

We recall that a groupoid is category with all morphisms invertible; a groupoid is essentially finite if it is equivalent to a groupoid with a finite number of morphism. The map  $|\cdot|_g$  is invariant under equivalence of groupoids and is such that

$$|\emptyset| = 0, \quad |[1]| = 1, \quad |G \sqcup H| = |G| + |H|, \quad \text{and} \quad |G \times H| = |G||H|.$$

Another useful property of  $|\cdot|_g$  is the following. Assume a finite group  $G$  acts on the finite set  $X$ , then we let  $X \rtimes G$  be the groupoid with set of objects  $X$  and such that

$$X \rtimes G(x, y) = \{g \in G \mid gx = y\}.$$

It is easy to check that

$$|X \rtimes G|_g = \frac{|X|}{|G|}.$$

Note that  $\overline{X \rtimes G}$  is the quotient set  $X/G$ .

Our immediate goal is to understand the Möbius function for isocyclic essentially locally finite categories in terms of the cardinality of groupoids, so we fix one of those categories  $C$ . For  $\bar{x} \in \overline{C}$ , the set of morphisms  $C(x, x)$  from  $x$  to itself is a group, moreover the group  $C(x, x) \times C(x, x)$  acts on  $C(x, x)$  by pre and post composition of morphisms. We have that

$$\mu[\bar{x}, \bar{x}] = \frac{1}{|C(x, x)|} = \frac{|C(x, x)|}{|C(x, x)|^2} = |C(x, x) \rtimes (C(x, x) \times C(x, x))|_g.$$

For  $\bar{x} < \bar{y}$  in  $\overline{C}$ , consider the groupoid  $[[0, n], C]_{\bar{x}, \bar{y}}^{\natural}$  (the odd notation will be justified below) whose objects are functors

$$F : [0, n] \longrightarrow C$$

from the interval  $[0, n] \subseteq \mathbb{N}$  to  $C$  such that  $F(0) \simeq x$ ,  $F(n) \simeq y$ , and  $F(i < i + 1)$  is not an isomorphism for  $i \in [0, n - 1]$ .

Morphisms in  $[[0, n], C]_{\bar{x}, \bar{y}}^{\natural}$  are natural isomorphisms. Concretely, objects in  $[[0, n], C]_{\bar{x}, \bar{y}}^{\natural}$  are diagrams in  $C$  of the form

$$x \simeq x_0 \longrightarrow x_1 \longrightarrow \dots \longrightarrow x_n \simeq y,$$

where none of the arrows is an isomorphism. Morphisms in  $[[0, n], C]_{\bar{x}, \bar{y}}^{\natural}$  are commutative diagrams

$$\begin{array}{ccccccc} x_0 & \longrightarrow & x_1 & \longrightarrow & \dots & \longrightarrow & x_{n-1} & \longrightarrow & x_n \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ z_0 & \longrightarrow & z_1 & \longrightarrow & \dots & \longrightarrow & z_{n-1} & \longrightarrow & z_n \end{array}$$

where the vertical arrows are isomorphisms.

Let us compute the cardinality of the groupoid  $[[0, n], C]_{\bar{x}, \bar{y}}^{\natural}$ . Note that

$$[[0, n], C]_{\bar{x}, \bar{y}}^{\natural} = \bigsqcup_{\bar{x} \simeq \bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_{n-1} < \bar{x}_n \simeq \bar{y}} [[0, n], C]_{\bar{x}, \bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y}}$$

where the groupoids  $[[0, n], C]_{\bar{x}, \bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y}}$  are defined just as  $[[0, n], C]_{\bar{x}, \bar{y}}$  fixing beforehand the isomorphism classes of the intermediate objects  $\bar{x}_1, \dots, \bar{x}_{n-1}$ . The groupoids

$$[[0, n], C]_{\bar{x}, \bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y}}$$

are actually quite easy to understand. Objects and morphism in it are isomorphic to diagrams of the form

$$\begin{array}{ccccccc} x & \xrightarrow{f_1} & x_1 & \xrightarrow{f_2} & \dots & \xrightarrow{f_{n-1}} & x_{n-1} & \xrightarrow{f_n} & x_n \\ g_0 \downarrow & & g_1 \downarrow & & & & g_{n-1} \downarrow & & g_n \downarrow \\ x & \xrightarrow{g_1 f_1 g_0^{-1}} & x_1 & \xrightarrow{g_2 f_2 g_1^{-1}} & \dots & \xrightarrow{g_{n-1} f_{n-1} g_{n-2}^{-1}} & x_{n-1} & \xrightarrow{g_n f_n g_{n-1}^{-1}} & x_n \end{array}$$

for which the top horizontal arrows together with the vertical arrows uniquely determine the bottom arrows.

From this viewpoint is clear that we have an equivalence of groupoids

$$[[0, n], C]_{\bar{x}, \bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y}} \simeq \left( \prod_{i=0}^{n-1} C(x_i, x_{i+1}) \right) \rtimes \left( \prod_{i=0}^n C(x_i, x_i) \right),$$

and thus we have that

$$\left| [[0, n], C]_{\bar{x}, \bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y}} \right|_g = \frac{|C(x_0, x_1)| \dots |C(x_{n-1}, x_n)|}{|C(x_0, x_0)| |C(x_1, x_1)| \dots |C(x_{n-1}, x_{n-1})| |C(x_n, x_n)|}$$

and we have obtained the following result.

**Lemma 63.** Let  $C$  be an isocyclic essentially locally finite category and  $\bar{x} < \bar{y}$  in  $\overline{C}$ . We have that:

$$\mu[\bar{x}, \bar{y}] = \sum_{n \geq 1} \sum_{\bar{x} = \bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_n = \bar{y}} (-1)^n \left| [[0, n], C]_{\bar{x}, \bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y}} \right|_g = \sum_{n \geq 1} (-1)^n \left| [[0, n], C]_{\bar{x}, \bar{y}}^{\natural} \right|_g.$$

*Proof.* Follows from Theorem 60 and the formula above.  $\square$

For our next constructions we need a few notions from the theory of simplicial sets [14, 27, 34]. We are going to show that the formula above for  $\mu[\bar{x}, \bar{y}]$  can be understood in terms of augmented simplicial essentially finite groupoids. A simplicial essentially finite groupoid is a functor

$$G : \Delta^\circ \longrightarrow \text{gpd},$$

where  $\Delta^\circ$  is the opposite category of  $\Delta$ . Objects in  $\Delta$  are the intervals  $[0, n] \subseteq \mathbb{N}$  for  $n \in \mathbb{N}$ . Morphisms in  $\Delta$  are non-decreasing maps.

Thus a simplicial essentially finite groupoid  $G$  assigns an essentially finite groupoid  $G_n$  to each  $n \in \mathbb{N}$ , and a functor

$$\widehat{f} : G_m \longrightarrow G_n$$

to each non-decreasing map  $f : [0, n] \longrightarrow [0, m]$ .

An augmented simplicial essentially finite groupoid is a functor

$$G : \Delta_a^\circ \longrightarrow \text{gpd}$$

where  $\Delta_a$  is the category obtained from  $\Delta$  by adjoining an object  $[-1]$  and a unique morphism  $[-1] \longrightarrow [0, n]$  for each  $n \geq -1$ . Thus an augmented simplicial groupoid  $G$  has, in addition, a groupoid  $G_{-1}$  and a unique functor  $G_n \longrightarrow G_{-1}$  for each  $n \geq -1$ .

For  $G$  an augmented simplicial groupoid and  $n \geq -1$ , we let  $G_n^\natural$  be the full subgroupoid of  $G_n$  whose objects are the non-degenerated objects of  $G_n$ . An object  $x \in G_n$  is called degenerated if there is a non-decreasing map  $f : [0, n] \longrightarrow [0, m]$  with  $m < n$ , and an object  $y \in G_m$  such that  $\widehat{f}(y) \simeq x$ .

**Definition 64.** The reduced Euler characteristic  $\widetilde{\chi}_g G$  of an augmented simplicial groupoid  $G : \Delta_a^\circ \longrightarrow \text{gpd}$  is given by

$$\widetilde{\chi}_g G = \sum_{n \geq -1} (-1)^n |G_n^\natural|_g.$$

Let  $C$  be an isocyclic essentially locally finite category. For  $\overline{x} < \overline{y}$  in  $\overline{C}$ , we define the augmented simplicial groupoid  $C_*(\overline{x}, \overline{y})$  as follows. For  $n \geq -1$ , we set

$$C_n(\overline{x}, \overline{y}) = [[0, n+2], C]_{\overline{x}, \overline{y}}.$$

Given a morphism  $f : [0, n] \longrightarrow [0, m]$  in  $\Delta$ , consider the non-decreasing extension map  $f_e$  defined through the commutative diagram

$$\begin{array}{ccc} [0, n] & \xrightarrow{f} & [0, m] \\ \downarrow & & \downarrow \\ \{0\} \sqcup [1, n+1] \sqcup \{n+2\} & \xrightarrow{f_e} & \{0\} \sqcup [1, m+1] \sqcup \{m+2\} \end{array}$$

where the vertical arrows arise from the increasing bijections

$$[0, n] \longrightarrow [1, n+1] \quad \text{and} \quad [0, m] \longrightarrow [1, m+1],$$

and we set  $f_e(0) = 0$ ,  $f_e(n+2) = m+2$ . Using this notation the functor

$$\widehat{f} : [[0, m+2], C]_{\overline{x}, \overline{y}} \longrightarrow [[0, n+2], C]_{\overline{x}, \overline{y}}$$

is given by

$$\widehat{f}(F) = Ff_e.$$

If we have another morphism  $g : [0, m] \longrightarrow [0, k]$  in  $\Delta$ , then we have that

$$(gf)_e = g_e f_e \quad \text{and thus}$$

$$\widehat{gf}(F) = F(gf)_e = F(g_e f_e) = (Fg_e)f_e = (\widehat{f}\widehat{g})(F).$$

The required unique functor  $[[0, m+2], C]_{\overline{x}, \overline{y}} \longrightarrow [[0, 1], C]_{\overline{x}, \overline{y}}$  sends a diagram

$$\begin{array}{ccccccc} x_0 & \xrightarrow{f_1} & x_1 & \xrightarrow{f_2} & \dots & \xrightarrow{f_{n+1}} & x_{n+1} & \xrightarrow{f_{n+2}} & x_{n+2} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ z_0 & \xrightarrow{g_1} & z_1 & \xrightarrow{g_2} & \dots & \xrightarrow{g_{n+1}} & z_{n+1} & \xrightarrow{g_{n+2}} & z_{n+2} \end{array}$$

to the diagram

$$\begin{array}{ccc} x_0 & \xrightarrow{f_{n+2}f_{n+1}\dots f_2f_1} & x_{n+2} \\ \downarrow & & \downarrow \\ z_0 & \xrightarrow{g_{n+2}g_{n+1}\dots g_2g_1} & z_{n+2} \end{array}$$

Thus  $C_*(\overline{x}, \overline{y})$  is an augmented simplicial groupoid. All together we have shown the following result.

**Theorem 65.** Let  $C$  be an isocyclic essentially locally finite category and  $\overline{x} < \overline{y}$  in  $\overline{C}$ . Then

$$\mu[\overline{x}, \overline{y}] = \widetilde{\chi}_g C_*(\overline{x}, \overline{y}).$$

*Proof.* By Lemma 63 we have that

$$\begin{aligned} \mu[\overline{x}, \overline{y}] &= \sum_{n \geq 1} (-1)^n \left| [[0, n], C]_{\overline{x}, \overline{y}}^{\natural} \right|_g = \\ &= \sum_{n \geq -1} (-1)^n \left| [[0, n+2], C]_{\overline{x}, \overline{y}}^{\natural} \right|_g = \sum_{n \geq -1} (-1)^n \left| C_n(\overline{x}, \overline{y}) \right|_g = \widetilde{\chi}_g C_*(\overline{x}, \overline{y}). \end{aligned}$$

□

Next we consider another distinguished element  $\xi_g$  in the incidence algebra  $[\mathbb{I}(\overline{C}, \leq), R]$  of an isocyclic essentially locally finite category  $C$  given for  $\overline{x} \leq \overline{y} \in \overline{C}$  by:

$$\xi_g[\overline{x}, \overline{y}] = \left| C(x, y) \rtimes (C(x, x) \times C(y, y)) \right|_g.$$

Thus we have that

$$\xi_g[\bar{x}, \bar{y}] = \frac{|C(x, y)|}{|C(x, x)||C(y, y)|} = \frac{\xi[\bar{x}, \bar{y}]}{|C(x, x)||C(y, y)|}.$$

For  $\bar{x} < \bar{y} \in \bar{C}$ , we let  $D_*(x, y)$  be the augmented sub-simplicial groupoid of  $C_*(\bar{x}, \bar{y})$  such that for  $n \geq -1$ :

- Objects in  $D_n(x, y)$  are functors  $F : [0, n+2] \longrightarrow C$  with  $F(0) = x$  and  $F(n+2) = y$ .
- A morphism in  $D_n(x, y)$  from  $F$  to  $G$  is a natural isomorphism  $l : F \longrightarrow G$  such that the morphisms  $l(0) : x \longrightarrow x$  and  $l(n+2) : y \longrightarrow y$  are identities.

Thus a morphism in  $D_*(x, y)$  is a commutative diagram of the form

$$\begin{array}{ccccccc} & & x_1 & \longrightarrow & x_2 & \longrightarrow & \dots & \longrightarrow & x_n & \longrightarrow & x_{n+1} & & \\ & \nearrow & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \searrow & \\ x & & z_1 & \longrightarrow & z_2 & \longrightarrow & \dots & \longrightarrow & z_n & \longrightarrow & z_{n+1} & & \\ & \searrow & & & & & & & & & & \nearrow & \\ & & & & & & & & & & & & y \end{array}$$

where the diagonal and horizontal arrows are not allowed to be isomorphisms, and the vertical arrows are isomorphisms.

**Theorem 66.** The map  $\xi_g$  is a unit in  $[\mathbb{I}(\bar{C}_{\leq}), R]$  and its inverse  $\mu_g$  is given for  $\bar{x} < \bar{y}$  in  $\bar{C}$  by

$$\mu_g[\bar{x}, \bar{x}] = |C(x, x)|, \quad \text{and}$$

$$\mu_g[\bar{x}, \bar{y}] = \sum_{n \geq 1} \sum_{\bar{x} = \bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_n = \bar{y}} (-1)^n \frac{|C(x_0, x_1)| \dots |C(x_{n-1}, x_n)|}{|C(x_1, x_1)| \dots |C(x_{n-1}, x_{n-1})|}.$$

Moreover, we have that:

$$\mu_g[\bar{x}, \bar{y}] = \tilde{\chi}_g D_*(\bar{x}, \bar{y}).$$

*Proof.* It follows from Theorem 15 that:

$$\begin{aligned} \mu_g[\bar{x}, \bar{y}] &= \sum_{n \geq 1} \sum_{\bar{x} = \bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_n = \bar{y}} (-1)^n \frac{\xi_g[x_0, x_1] \dots \xi_g[x_{n-1}, x_n]}{\xi_g[x_0, x_0] \xi_g[x_1, x_1] \dots \xi_g[x_{n-1}, x_{n-1}] \xi_g[x_n, x_n]} = \\ &= \sum_{n \geq 1} \sum_{\bar{x} = \bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_n = \bar{y}} (-1)^n \frac{\frac{|C(x_0, x_1)|}{|C(x_0, x_0)||C(x_1, x_1)|} \dots \frac{|C(x_{n-1}, x_n)|}{|C(x_{n-1}, x_{n-1})||C(x_n, x_n)|}}{\frac{|C(x_0, x_0)|}{|C(x_0, x_0)||C(x_0, x_0)|} \dots \frac{|C(x_n, x_n)|}{|C(x_n, x_n)||C(x_n, x_n)|}} = \\ &= \sum_{n \geq 1} \sum_{\bar{x} = \bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_n = \bar{y}} (-1)^n \frac{|C(x_0, x_1)| \dots |C(x_{n-1}, x_n)|}{|C(x_1, x_1)| \dots |C(x_{n-1}, x_{n-1})|}. \end{aligned}$$

□

## 7 Möbius Categories

In this and the next section we adopt the viewpoint that regards  $\mathbb{N}_+$ , in the classical Möbius theory, as a monoid. The main idea is to develop a Möbius theory that applies to a large class of monoids. It turns out that this goal can be readily achieved for the category of finite decomposition monoids. Indeed one can go further and define a Möbius theory that applies to finite decomposition categories, better known in the literature as Möbius categories. Since a monoid is just a category with one object, the theory of finite decomposition monoids embeds into the theory of Möbius categories. The notion of Möbius categories was introduced by Leroux in [10, 24], and since then there have been quite a few publications in the field, among them [15, 20, 23, 33]. Here we limit ourselves to the most basic results on Möbius categories.

Let  $C$  be a category and  $f$  a morphism in  $C$ . A  $n$ -decomposition of  $f$ , for  $n \geq 1$ , is  $n$ -tuple  $(f_1, \dots, f_n)$  of morphisms in  $C$  such that

$$f_n \dots f_1 = f.$$

Thus a  $n$ -decomposition of a morphism  $f : x \rightarrow y$  is a commutative diagram of the form

$$\begin{array}{ccccccc} & & x_1 & \xrightarrow{f_2} & x_2 & \xrightarrow{f_3} & \dots & \xrightarrow{f_{n-2}} & x_{n-1} & \xrightarrow{f_{n-1}} & x_{n-1} & \searrow f_n \\ & \nearrow f_1 & & & & & & & & & & \\ x & & & & & & & & & & & y \\ & & & & & & & & & & \nearrow f & \end{array}$$

Let  $D_n f$  be the set of  $n$ -decompositions of  $f$ . A decomposition is called proper if none of its components is an identity morphism. For  $n \geq 2$ , let  $PD_n f$  be the set of proper  $n$ -decompositions of  $f$ . Set  $PD_1 f = D_1 f = \{f\}$  and

$$Df = \bigsqcup_{n \geq 1} D_n f \quad \text{and} \quad PDf = \bigsqcup_{n \geq 1} PD_n f.$$

**Definition 67.** A category  $C$  is Möbius if  $PDf$  is a finite set for all morphisms  $f$  in  $C$ , i.e. each morphism in  $C$  admits a finite number of proper decompositions.

Note that in a Möbius category the only isomorphisms are the identities.

**Lemma 68.** If  $PDf$  is a finite set, then  $D_n f$  is a finite set for all  $n \geq 1$ . Indeed we have:

$$|D_n f| = \sum_{k \geq 1}^n \binom{n}{k} |PD_k f| \quad \text{and} \quad |PD_n f| = \sum_{k \geq 1}^n (-1)^{n-k} \binom{n}{k} |D_k f|.$$

*Proof.* The left identity is shown as follows. Out of the  $n$  morphisms in a  $n$ -decomposition assume that  $n - k$  are identities which can be placed in  $\binom{n}{k}$  different positions. Omitting the identity morphisms, a  $n$ -decomposition of the latter type reduces to a proper  $k$ -decomposition. The right identity follows from the Möbius inversion formula.  $\square$



Let  $f$  be a morphism in a category  $C$ . We say that  $f$  fixes a morphism  $g \in C$  if

$$fg = g \quad \text{or} \quad gf = g.$$

Next result is due to Leroux [10, 20].

**Theorem 69.** A category  $C$  is Möbius if and only if the following conditions hold:

- $\text{PD}_2 f$  is a finite set for each morphism  $f$  in  $C$ .
- Identities admit no proper decomposition.
- If  $f$  fixes a morphism, then  $f$  is an identity morphism.

*Proof.* Assume  $C$  is Möbius category. By definition  $\text{PD}_2 f$  is a finite set for each  $f \in C$ . If an identity morphism  $1$  in  $C$  admits a proper decomposition

$$f_n \dots f_1 = 1,$$

then it admits infinitely many proper decompositions, indeed

$$1 = f_n \dots f_1 = f_n \dots f_1 f_n \dots f_1 = f_n \dots f_1 f_n \dots f_1 f_n \dots f_1 = \dots$$

Let  $f$  be a non-identity morphism and  $g$  be another morphism. If  $fg = g$ , then  $g$  is a non-identity morphism that admits infinitely many proper decompositions

$$g = fg = ffg = fffg = fffg = \dots$$

Thus we conclude that  $fg \neq g$ , and a similar argument shows that  $gf \neq g$  as well.

Suppose now that the three conditions of the theorem hold. An inductive argument shows that if  $\text{PD}_2 f$  is a finite set, then  $\text{PD}_n f$  is a finite set for all  $n \geq 2$ . Let  $k = |\text{PD}_2 f|$ , we show that  $f$  can not have a proper decomposition of length greater or equal  $k + 2$ . Assume that we have such a decomposition  $(f_1, \dots, f_{k+2})$ . Composing initial and final segments of morphisms in  $(f_1, \dots, f_{k+2})$  one obtains  $k + 1$  proper 2-decompositions of  $f$ . They can not be all different, thus there exist  $g$  and  $h$ , obtained by composing some of the  $f_i$ , such that  $gh = h$ . Then  $h$  must be an identity, in contradiction with the fact that identities admit no proper decomposition.  $\square$

**Definition 70.** A  $\mathbb{N}$ -graded category with finite graded components  $C$  is a category such that for  $x, y, z \in C$  we have

$$C(x, y) = \bigsqcup_{n \in \mathbb{N}} C_n(x, y),$$

$$C_0(x, x) = \{1_x\}, \quad |C_n(x, y)| < \infty \quad \text{and} \quad C_n(x, y) \times C_m(y, z) \longrightarrow C_{n+m}(x, z).$$

The degree  $\deg f \in \mathbb{N}$  of a morphism  $f : x \longrightarrow y$  is such that  $f \in C_{\deg f}(x, y)$ .

**Example 71.** Let  $\Gamma$  be a finite directed graph. Let  $P_\Gamma$  be the category of paths in  $\Gamma$ , i.e. objects of  $P_\Gamma$  are the vertices of  $\Gamma$ , and morphisms in  $P_\Gamma(x, y)$  are paths in  $\Gamma$  from  $x$  to  $y$ . By convention there is an empty path of length 0 from each vertex to itself. Let  $R$  be an equivalence relation on  $\Gamma$ -paths generated by a collection of pairs of paths in  $\Gamma$  of the same length and with the same endpoints. Let  $P_R$  be the category whose objects are the vertices of  $\Gamma$  and whose morphisms are equivalence classes of paths in  $\Gamma$ . The category  $P_R$  is  $\mathbb{N}$ -graded with finite graded components.

**Proposition 72.** Let  $C$  be a  $\mathbb{N}$ -graded category with finite graded components, let  $(X, \leq)$  be a locally finite poset, and  $F : X \rightarrow C$  be a functor. Let  $X_F$  be the category with  $X$  as its set of objects and with morphisms given by

$$X_F(x, y) = \begin{cases} C(F(x), F(y)) & \text{if } x \leq y, \\ \emptyset & \text{otherwise.} \end{cases}$$

The category  $X_F$  is Möbius.

*Proof.* Fix  $x, y \in C$ . Note that  $X_F(x, y) \neq \emptyset$  if and only if  $x \leq y$  as elements of  $X$ . Since  $(X, \leq)$  is a locally finite poset there is only a finite number of choices of objects  $x_1, \dots, x_{n-1} \in C$  for which there is a diagram

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} y \quad \text{with} \quad f_n \dots f_1 = f.$$

Note that the latter identity implies that

$$\deg f_1 + \deg f_2 + \dots + \deg f_n = \deg f,$$

so given the tuple  $x_1, \dots, x_{n-1}$  there is only a finite number of diagrams as the above.  $\square$

**Definition 73.** A category  $C$  is one way if in any diagram  $x \xrightarrow{f} y \xrightarrow{g} x$  in  $C$ , we have that  $x = y$  and  $f = g = 1_x$ .

**Proposition 74.** Let  $C$  be a category with  $C(x, y)$  a finite set for all  $x, y \in C$ . Then  $C$  is a Möbius category if and only if  $C$  is a locally finite and one way category.

*Proof.* Assume  $C$  is a locally finite category. Objects of  $C$  are partially ordered according to Lemma 48. For  $x \leq y$  in  $C$  we let  $C[x, y]$  be the set of linearly ordered subsets of the interval  $[x, y]$ . The map  $\text{PDF} : C[x, y] \rightarrow C[x, y]$  sending  $(f_1, \dots, f_n)$  to the linearly ordered set

$$\{tf_1 < tf_2 < \dots < tf_{n-1}\} \subseteq [x, y]$$

has a finite codomain since  $[x, y]$  is a finite set as  $C$  is locally finite, it has finite fibers because morphism between objects of  $C$  are finite sets, thus  $\text{PDF}$  is a finite set.

Assume now that  $C$  is a Möbius category with finite sets of morphisms. The map

$$\bigsqcup_{f \in C(x,y)} \text{PD}f \longrightarrow C[x,y]$$

is surjective and has a finite domain. Thus the interval  $[x,y]$  is a finite set. Next we show that any endomorphism  $f : x \longrightarrow x$  in  $C$  is an identity. Since  $C(x,x)$  is a finite set, there are integers  $n \geq 0$  and  $k \geq 1$  such

$$f^n = f^{n+k} = f^n f^k.$$

So  $f^k$  fixes a morphism, and then  $f^k = 1$  by Theorem 69. Since 1 admits no proper decomposition we have that  $f = 1$ .  $\square$

Next we associate a convolution algebra to each category  $C$  such that  $D_2f$  is a finite set for all morphisms  $f$  in  $C$ .

**Definition 75.** Let  $C$  be a category such that  $D_2f$  is a finite set for all morphism  $f \in C_1$ . The convolution algebra of  $C$  is the pair  $([C_1, R], \star)$  where for  $\alpha, \beta \in [C_1, R]$  the product  $\alpha \star \beta : C_1 \longrightarrow R$  is given on  $f \in C_1$  by

$$\alpha \star \beta(f) = \sum_{(f_1, f_2) \in D_2f} \alpha(f_1)\beta(f_2).$$

**Proposition 76.**  $([C_1, R], \star)$  is an associative algebra with unit the map  $1 \in [C_1, R]$  such that  $1(f) = 1$  if  $f$  is an identity morphism, and zero otherwise.

*Proof.* Since  $D_2f$  is a finite set, then  $D_nf$  is a finite set for all  $n \geq 2$ . The unit property is clear. Associativity follows from the identities

$$\alpha_1 \star (\alpha_2 \star \alpha_3)(f) = \sum_{(f_1, f_2, f_3) \in D_3f} \alpha_1(f_1)\alpha_2(f_2)\alpha_3(f_3) = (\alpha_1 \star \alpha_2) \star \alpha_3(f).$$

More generally we have that

$$\alpha_1 \star \alpha_2 \star \dots \star \alpha_n(f) = \sum_{(f_1, \dots, f_n) \in D_nf} \alpha_1(f_1)\alpha_2(f_2)\dots\alpha_n(f_n).$$

$\square$

**Corollary 77.** Let  $C$  be a category with  $D_2f$  is a finite set for all morphism  $f \in C_1$ . The free  $R$ -module  $\langle C_1 \rangle$  generated by  $C_1$  together with the  $R$ -linear maps

$$\Delta : \langle C_1 \rangle \longrightarrow \langle C_1 \rangle \otimes \langle C_1 \rangle \quad \text{and} \quad \epsilon : \langle C_1 \rangle \longrightarrow R$$

given on generators, respectively, by

$$\Delta f = \sum_{(f_1, f_2) \in D_2f} f_1 \otimes f_2 \quad \text{and} \quad \epsilon f = \begin{cases} 1 & \text{if } f \text{ is an identity,} \\ 0 & \text{if } x \neq y, \end{cases}$$

is a  $R$ -coalgebra.

**Theorem 78.** Let  $C$  be a Möbius category. A map  $\alpha \in [C_1, R]$  is a  $\star$ -unit if and only if  $\alpha(1_x)$  is a unit in  $R$  for all  $x \in C$ . Thus the map  $\xi \in [C_1, R]$  constantly equal to 1 is a unit in  $([C_1, R], \star)$ ; its inverse  $\mu$ , called the Möbius function of  $C$ , is such that  $\mu(1_x) = 1$  for all  $x \in C$ , and if  $f$  is a non-identity morphism in  $C$  then we have that

$$\mu f = \sum_{n \geq 1} (-1)^n |\text{PD}_n f|.$$

*Proof.* Let  $\alpha, \beta \in [C_1, R]$  be such that  $\alpha \star \beta = 1$ . Then for all  $x \in C$  we have that

$$\alpha(1_x)\beta(1_x) = (\alpha \star \beta)(1_x) = 1, \quad \text{thus } \alpha(1_x) \text{ is a unit.}$$

Conversely, assume that  $\alpha(1_x)$  is a unit and write  $\alpha^{-1}(1_x) = \frac{1}{\alpha(1_x)}$ . Let  $f$  be a non-isomorphism in  $C$ , we have that

$$\alpha^{-1} f = \sum_{n \geq 1} \sum_{(f_1, \dots, f_n) \in \text{PD}_n f} (-1)^n \frac{\alpha(f_1) \dots \alpha(f_n)}{\alpha(1_{sf_1}) \alpha(1_{tf_1}) \dots \alpha(1_{tf_n})},$$

where  $s, t : C_1 \rightarrow C_0$  are, respectively, the source and target maps of  $C$ . □

**Corollary 79.** Let  $C$  be a Möbius category and let  $R[[x_f]]$  be the  $R$ -algebra of formal power series in the variables  $x_f$  with  $f$  a non-identity in  $C_1$ . The structural maps on  $R[[x_f]]$  given, respectively, on generators by

$$\begin{aligned} \epsilon 1 &= 1 \quad \text{and} \quad \epsilon x_f = 0, \\ \Delta 1 &= 1 \otimes 1 \quad \text{and} \quad \Delta x_f = 1 \otimes x_f + \sum_{(f_1, f_2) \in \text{D}_2 f} x_{f_1} \otimes x_{f_2} + x_f \otimes 1, \\ S 1 &= 1 \quad \text{and} \quad S x_f = \sum_{n \geq 1} \sum_{(f_1, \dots, f_n) \in \text{PD}_n f} (-1)^n x_{f_1} \dots x_{f_n} \end{aligned}$$

turn  $R[[x_f]]$  into a Hopf algebra, and the Möbius  $\mu \in [C_1, R]$  of  $C$  is given by

$$\mu f = S x_f(1).$$

.

**Proposition 80.** Let  $C$  and  $D$  be finite Möbius categories, then  $C \times D$  is a finite Möbius category,  $(C \times D)_1 \simeq C_1 \times D_1$ , and we have a natural isomorphism of algebras

$$([C \times D]_1, \star) \simeq ([C_1, R], \star) \otimes ([D_1, R], \star).$$

Moreover, we have that  $\xi_{C \times D}(f, g) = \xi_C f \xi_D g$  and thus

$$\mu_{C \times D}(f, g) = \mu_C f \mu_D g.$$

Given a non-identity morphism  $f : x \longrightarrow y$  in  $C$ , we construct the augmented simplicial set  $D_*f$ , a variant of the classical bar resolution, as follows. For  $n \geq -1$  we have inclusions

$$D_n f \subseteq [[0, n+2], C]_{x,y},$$

where a functor  $F \in [[0, n+2], C]_{x,y}$  is in  $D_n f$  if and only if

$$F(0 \leq n+2) = f.$$

With the notation the decomposition of morphism of Definition 67, we have that:

$$D_n f = D_{n+2} f \quad \text{and} \quad D_n f^\natural = \text{PD}_{n+2} f.$$

**Theorem 81.** The Möbius function  $\mu \in [C_1, R]$  of a Möbius category  $C$  is given on a non-isomorphism  $f$  in  $C$  by

$$\mu f = \tilde{\chi} D_* f = \sum_{n \geq 0} (-1)^n \text{rank} \tilde{H}_n |D_* f|.$$

*Proof.* By Theorem 78 that

$$\mu f = \sum_{n \geq 1} (-1)^n |\text{PD}_n f| = \sum_{n \geq 0} (-1)^n \text{rank} \tilde{H}_n |D_* f|.$$

From this identity and the previous remarks we get that

$$\mu f = \sum_{n \geq 1} (-1)^n |\text{PD}_n f| = \sum_{n \geq -1} (-1)^n |\text{PD}_{n+2} f| = \sum_{n \geq -1} (-1)^n |D_n f^\natural| = \tilde{\chi} D_* f.$$

□

**Corollary 82.** The reduced homology groups  $\tilde{H}_n |D_* f|$  of the space  $|D_* f|$  are obtained from the differential complex over  $\mathbb{Z}$  such that:

- It is generated in degree  $n$  by the  $(n+2)$ -decompositions  $[f_1, \dots, f_{n+2}]$  of  $f$ .
- The differential  $d$  is given on generators by:

$$d[f] = 0 \quad \text{in degree } -1,$$

and it is given in degree  $n \geq 0$  by

$$d[f_1, \dots, f_{n+2}] = \sum_{k=1}^{n+1} (-1)^k [f_1, \dots, f_{k+1} f_k, \dots, f_{n+2}].$$

We close this section with a brief discussion of the relation between the Möbius theory for posets and the Möbius theory for finite decomposition categories.

**Proposition 83.** Let  $C$  be a Möbius category. The relation  $\leq$  on  $C_1$  given by

$$f \leq g \text{ if and only if there is morphism } h \in C_1 \text{ such that } g = hf,$$

is a partial order on  $C_1$ .

*Proof.* Reflexivity and transitivity are valid for arbitrary categories. If  $f \leq g \leq f$ , then there are morphisms  $h$  and  $k$  in  $C_1$  such that  $g = hf$  and  $f = kg$ . Thus  $f = (kh)f$ , and since  $C$  is Möbius and  $kh$  fixes  $f$ , then  $kh$  is an identity 1 in  $C$ . Moreover, since identities in  $C$  can not be properly decomposed, then  $k = h = 1$  by Theorem 69, and thus  $f = g$ .  $\square$

**Definition 84.** A category  $C$  is right cancellative if for any morphisms  $f, g, h$  in  $C$  we have:

$$gf = hf \quad \text{implies that} \quad g = h.$$

**Theorem 85.** Let  $C$  be a cancellative Möbius category. The convolution algebra  $[C_1, R]$  is isomorphic to the subalgebra

$$[\mathbb{I}_{(C_1, \leq)}, R]_c \subseteq [\mathbb{I}_{(C_1, \leq)}, R]$$

consisting of maps

$$\alpha : \mathbb{I}_{(C_1, \leq)} \longrightarrow R \quad \text{such that} \quad \alpha[f, gf] = \alpha[1, g] \quad \text{for all morphisms } f, g \in C_1.$$

*Proof.* Consider the map  $[C_1, R] \longrightarrow [\mathbb{I}_{(C_1, \leq)}, R]$  sending a map  $\beta : C_1 \longrightarrow R$  to the map  $\widehat{\beta} : \mathbb{I}_{(C_1, \leq)} \longrightarrow R$  given by

$$\widehat{\beta}[f, hf] = \beta(h).$$

The map  $\widehat{\beta}$  is well-defined since  $C$  is cancellative and thus  $f \leq g$  if and only if there exists an unique morphism  $h = \frac{f}{g}$  such that  $g = hf$ . It is clear that the image of  $\widehat{\beta}$  is included in  $[\mathbb{I}_{(C_1, \leq)}, R]_c$ , we show that is actually equal to  $[\mathbb{I}_{(C_1, \leq)}, R]_c$ . For  $\alpha \in [\mathbb{I}_{(C_1, \leq)}, R]_c$ , let  $\beta \in [C_1, R]$  be given by  $\beta(f) = \alpha[1, f]$ . Then we have that

$$\widehat{\beta}[f, hf] = \beta(h) = \alpha[1, h] = \alpha[f, hf], \quad \text{that is} \quad \widehat{\beta} = \alpha.$$

On the other hand assume that  $\widehat{\beta}_1 = \widehat{\beta}_2$ , then

$$\beta_1(h) = \widehat{\beta}_1[1, h] = \widehat{\beta}_2[1, h] = \beta_2(h).$$

So the map  $\beta \longrightarrow \widehat{\beta}$  is injective. It remains to show that it is also an algebra map.

$$\begin{aligned} \widehat{\beta}_1 \widehat{\beta}_2[1, g] &= \sum_{1 \leq f \leq g} \widehat{\beta}_1[1, f] \widehat{\beta}_2[f, g] = \sum_{1 \leq f \leq g} \beta_1(f) \beta_2\left(\frac{f}{g}\right) = \\ &= \sum_{hf=g} \beta_1(f) \beta_2(h) = \beta_1 \beta_2(g) = \widehat{\beta_1 \beta_2}[1, g]. \end{aligned}$$

$\square$

## 8 Essentially Finite Decomposition Categories

The notion of Möbius categories discussed in the previous section is invariant under isomorphisms of categories, but fails to be invariant under equivalence of categories. Indeed, as we have already remarked Möbius categories are skeletal. In this section we study a variant notion for which this issue is circumvented.

Let  $C$  be a category and  $\overline{C}_1$  be the set of isomorphisms classes of morphisms in  $C$ . Recall that morphisms  $f, g \in C_1$  are isomorphic if they fit into a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \downarrow \\ z & \xrightarrow{g} & w \end{array}$$

where the vertical arrows are isomorphisms.

**Definition 86.** Fix  $\overline{f} \in \overline{C}_1$ . A  $n$ -decomposition of  $\overline{f}$ , for  $n \geq 1$ , is a  $n$ -tuple  $(f_1, \dots, f_n)$  of morphisms in  $C$  such that there are isomorphisms  $\alpha$  and  $\beta$  in  $C$  for which

$$f_n \dots f_1 = \beta^{-1} f \alpha,$$

that is a  $n$ -decomposition of  $\overline{f}$  is given by a commutative diagram of the form

$$\begin{array}{ccccccc} x_0 & \xrightarrow{f_1} & x_1 & \xrightarrow{f_2} & \dots & \xrightarrow{f_{n-1}} & x_{n-1} & \xrightarrow{f_n} & x_n \\ \alpha \downarrow & & & & & & & & \downarrow \beta \\ x & \xrightarrow{\quad f \quad} & & & & & & & y \end{array}$$

with  $\alpha$  and  $\beta$  isomorphisms.

Let  $D_n \overline{f}$  be the groupoid whose objects are  $n$ -decompositions of  $\overline{f}$ , and whose morphisms are commutative diagrams of the form

$$\begin{array}{ccccccc} x_0 & \xrightarrow{f_1} & x_1 & \xrightarrow{f_2} & \dots & \xrightarrow{f_{n-1}} & x_{n-1} & \xrightarrow{f_n} & x_n \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ z_0 & \xrightarrow{g_1} & z_1 & \xrightarrow{g_2} & \dots & \xrightarrow{g_{n-1}} & z_{n-1} & \xrightarrow{g_n} & z_n \end{array}$$

where the top and bottom of the diagram are  $n$ -decompositions of  $\overline{f}$  and the vertical arrows are isomorphisms.

**Definition 87.** For  $n \geq 2$ , a decomposition  $(f_1, \dots, f_n)$  of  $\bar{f}$  is called proper if none of the morphisms  $f_i$  is an isomorphism. For  $n \geq 2$ , we let  $\text{PD}_n \bar{f}$  be the full subgroupoid of  $\text{D}_n \bar{f}$  whose objects are proper decompositions. We set  $\text{PD}_1 \bar{f} = \text{D}_1 \bar{f}$  and

$$\text{D} \bar{f} = \bigsqcup_{n \geq 1} \text{D}_n \bar{f} \quad \text{and} \quad \text{PD} \bar{f} = \bigsqcup_{n \geq 1} \text{PD}_n \bar{f},$$

i.e.  $(\text{PD} \bar{f})$   $\text{D} \bar{f}$  is the groupoid of all (proper) decompositions of  $\bar{f}$ .

**Definition 88.** An essentially finite decomposition category  $C$  is a category such that  $\overline{\text{PD} \bar{f}}$  is a finite set for  $\bar{f} \in \overline{C}_1$ .

**Lemma 89.** If  $\overline{\text{PD} \bar{f}}$  is a finite set, then  $\overline{\text{D}_n \bar{f}}$  is a finite set for  $n \geq 1$  and we have that:

$$|\overline{\text{D}_n \bar{f}}| = \sum_{k \geq 1} \binom{n}{k} |\overline{\text{PD}_k \bar{f}}| \quad \text{and} \quad |\overline{\text{PD}_n \bar{f}}| = \sum_{k \geq 1} (-1)^{n-k} \binom{n}{k} |\overline{\text{D}_k \bar{f}}|.$$

*Proof.* The identity on the left is shown as follows. Out of the  $n$  morphisms in a  $n$ -decomposition assume that  $n - k$  are isomorphism, they can be placed in  $\binom{n}{k}$  different positions. Such a  $n$ -decomposition is isomorphic to a  $n$ -decomposition where the isomorphisms are replaced by identities and the remaining morphisms are not isomorphisms. Omitting the identity arrows a  $n$ -decomposition of the latter type reduces to a proper  $k$ -decomposition. The identity on the right follows from Möbius inversion formula.  $\square$

**Theorem 90.** Assume  $C$  is a category with  $C(x, y)$  finite for  $x, y \in C$ . Then  $C$  is an essentially finite decomposition category if and only if  $C$  is an isocyclic essentially locally finite category.

*Proof.* Let  $C$  be an essentially finite decomposition category. Let  $f : x \rightarrow x$  be an endomorphism in  $C$ , we show that  $f$  is an isomorphism. Since  $C(x, x)$  is a finite set and

$$\{1, f, f^2, \dots, f^n, \dots\} \subseteq C(x, x),$$

there must be  $n \geq 0$  and  $k \geq 1$  such that  $f^n = f^{n+k}$ . If  $n = 0$ , then  $f^k = 1$  for some  $k \geq 1$  and thus  $f$  is an isomorphism. If  $n > 0$  and  $f$  is not an isomorphism then the identities

$$f^n = f^{n+k} = f^{n+2k} = f^{n+3k} = \dots$$

show that  $f^n$  admits infinitely many decompositions, a contradiction. Thus in any case  $f$  has to be isomorphism.

Next we show that the intervals  $[\bar{x}, \bar{y}]$  in  $\overline{C}$ , with the order coming from Lemma 55, are finite sets. The map

$$\{\bar{x}, \bar{y}\} \sqcup \bigsqcup_{f \in C(x, y)} \overline{\text{PD}_2 \bar{f}} \longrightarrow [\bar{x}, \bar{y}],$$



being the identity on  $\{\bar{x}, \bar{y}\}$  and sending a 2-decomposition  $x \longrightarrow z \longrightarrow y$  of  $\bar{f}$  to  $\bar{z}$ , is surjective and has a finite domain because  $C$  is a finite decomposition category and  $C(x, y)$  is a finite set. Thus  $[\bar{x}, \bar{y}]$  is a finite set and  $C$  is an isocyclic essentially locally finite category.

Assume now that  $C$  is an isocyclic essentially locally finite category. Isomorphisms in  $C$  admit a unique decomposition modulo equivalences. Given a non-isomorphism  $f \in C(x, y)$  and a chain

$$\bar{x} < \bar{x}_1 < \dots < \bar{x}_{n-1} < \bar{y} \text{ in } \overline{C}, \quad \text{we let } \text{PD}(\bar{f}, \bar{x}_1, \dots, \bar{x}_{n-1})$$

be the full subcategory of  $\text{PD}\bar{f}$  whose objects are proper  $n$ -decompositions of  $f$  with the specified isomorphism classes for the intermediate objects. We have that

$$\text{PD}\bar{f} = \bigsqcup_{n \geq 2} \bigsqcup_{\bar{x} < \bar{x}_1 < \dots < \bar{x}_{n-1} < \bar{y}} \text{PD}(\bar{f}, \bar{x}_1, \dots, \bar{x}_{n-1}).$$

Since  $[\bar{x}, \bar{y}]$  is a finite set, it has a finite number of increasing chains. Since there are a finite number of morphisms between objects of  $C$ , the sets  $\overline{\text{PD}}(\bar{f}, \bar{x}_1, \dots, \bar{x}_{n-1})$  are finite. Therefore,  $\overline{\text{PD}}\bar{f}$  is a finite set, and  $C$  an essentially finite decomposition category.  $\square$

Next we associate a convolution algebra for a category  $C$  such that  $\overline{\text{D}}_2\bar{f}$  is a finite set for  $\bar{f} \in \overline{C}_1$ . The convolution product  $\star$  on  $[\overline{C}_1, R]$  is given on  $\alpha, \beta \in [\overline{C}_1, R]$  by

$$\alpha \star \beta(\bar{f}) = \sum_{(\bar{f}_1, \bar{f}_2) \in \overline{\text{D}}_2\bar{f}} \alpha(\bar{f}_1)\beta(\bar{f}_2).$$

Associativity for the product  $\star$  is by no means obvious. Thus we impose a (fairly strong) condition on  $C$  that guarantees associativity. In the midst of the proof of Theorem 95 below we provided a weaker (though less intuitive) condition that also guarantees associativity.

**Definition 91.** We say that a category  $C$  has the isomorphism filling property if any commutative diagram

$$\begin{array}{ccc} & y_1 & \\ x & \nearrow & \searrow y \\ & y_2 & \end{array}$$

with  $y_1$  and  $y_2$  isomorphic objects in  $C$ , can be enhanced to a commutative diagram

$$\begin{array}{ccc} & y_1 & \\ x & \nearrow & \searrow y \\ & \downarrow & \\ & y_2 & \end{array}$$

where the vertical arrow is an isomorphism.

**Example 92.** The category  $\mathbb{I}$  of finite sets and injective maps is isomorphism filling.

**Example 93.** The category  $\text{vect}$  of finite dimensional vector spaces and injective linear maps is isomorphism filling.

These previous examples follow from a general construction which we proceed to describe. Let  $\mathbb{B}$  be the category of finite sets and bijections,  $F : \mathbb{B} \rightarrow \text{set}$  a functor (a.k.a. a combinatorial species [1, 7, 11, 16]). Let  $\text{Par}_F : \mathbb{B} \rightarrow \text{set}$  be the species of  $F$ -colored partitions, i.e. for  $x \in \mathbb{B}$  we let  $\text{Par}_F x$  be the set of pairs  $(\pi, s)$  where  $\pi$  is a partition of  $x$  and  $s$  assigns to each  $b \in \pi$  an element  $s_b \in Fb$ . Consider the category  $\text{EPar}_F$  whose objects are triples  $(x, \pi, s)$  with  $(\pi, s) \in \text{Par}_F x$ . A morphism

$$f : (x, \pi, s) \rightarrow (z, \sigma, u) \quad \text{in } \text{EPar}_F$$

is an injective map  $f : x \rightarrow z$  such that there is a triple  $(y, \rho, t) \in \text{EPar}_F$  such that

$$(fx \sqcup y, f\pi \sqcup \rho, fs \sqcup t) = (z, \sigma, u).$$

**Lemma 94.** The category  $\text{EPar}_F$  is isomorphism filling.

*Proof.* Objects of  $\text{EPar}_F$  are triples  $(x, \pi, s)$  which we denote just by  $s$ , since the map  $s$  already includes the information about  $x$  and  $\pi$ . The category  $\text{EPar}_F$  is monoidal with disjoint union as product, and it is complemented in the sense that for any morphism  $f : s \rightarrow u$  there is a complement  $u \setminus_f s \in \text{EPar}_F$  such that

$$s \sqcup u \setminus_f s \simeq u.$$

Moreover, if we have morphisms  $s \rightarrow t_1$  and  $x \rightarrow t_2$ , with  $t_1$  and  $t_2$  isomorphic, then it follows that  $t_1 \setminus s$  and  $t_2 \setminus x$  are isomorphic objects and thus there is an isomorphism from  $t_1$  to  $t_2$  respecting the given morphisms.  $\square$

**Theorem 95.** Let  $C$  be an isomorphism filling category with  $\overline{D}_2 \overline{f}$  a finite set for  $\overline{f} \in \overline{C}_1$ . The product  $\star$  turns  $[\overline{C}_1, R]$  into an associative algebra with unit  $1 \in [\overline{C}_1, R]$  the map such that  $1(\overline{f}) = 1$  if  $f$  is an identity morphism, and zero otherwise.

*Proof.* The unit property is clear. Associativity follows from the identities

$$(\alpha_1 \star \alpha_2) \star \alpha_3(\overline{f}) = \sum_{(\overline{f}_1, \overline{f}_2, \overline{f}_3) \in \overline{D}_3 \overline{f}} \alpha_1(\overline{f}_1) \alpha_2(\overline{f}_2) \alpha_3(\overline{f}_3) = \alpha_1 \star (\alpha_2 \star \alpha_3)(\overline{f}).$$

We show the left-hand side identity, the right-hand side identity is proven similarly. By definition we have that

$$(\alpha_1 \star \alpha_2) \star \alpha_3(\overline{f}) = \sum_{(\overline{f}_1, \overline{f}_2) \in \overline{D}_2 \overline{f}} (\alpha_1 \star \alpha_2)(\overline{f}_1) \alpha_3(\overline{f}_2) =$$

$$\sum_{(\overline{f_1, f_2}) \in \overline{D_2 f}} \sum_{(\overline{f_3, f_4}) \in \overline{D_2 f_1}} \alpha_1(\overline{f_3}) \alpha_1(\overline{f_4}) \alpha_3(\overline{f_2}).$$

Setting

$$E = \left\{ (\overline{(f_1, f_2)}, \overline{(f_3, f_4)}) \mid \overline{(f_1, f_2)} \in \overline{D_2 f} \text{ and } \overline{(f_3, f_4)} \in \overline{D_2 f_1} \right\},$$

the desired result follows if there is a bijection  $\tau : \overline{D_3 f} \rightarrow E$  making the diagram

$$\begin{array}{ccc} \overline{D_3 f} & \xrightarrow{\tau} & E \\ & \searrow & \swarrow \\ & < \overline{C_1} > \otimes < \overline{C_1} > \otimes < \overline{C_1} > \end{array}$$

commutative, where the diagonal arrows are given, respectively, by

$$\begin{aligned} \overline{(f_1, f_2, f_3)} &\longrightarrow \overline{f_1} \otimes \overline{f_2} \otimes \overline{f_3}, \quad \text{and} \\ \overline{((f_1, f_2), (f_3, f_4))} &\longrightarrow \overline{f_3} \otimes \overline{f_4} \otimes \overline{f_2}. \end{aligned}$$

The desired map  $\tau : \overline{D_3 f} \rightarrow E$  is given by

$$\overline{(f_1, f_2, f_3)} \longrightarrow \overline{((f_2 f_1, f_3), (f_1, f_2))}.$$

Let us show that  $\tau$  is well-defined, surjective, and injective.

Let  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$  be isomorphisms from the appropriate objects so that

$$\overline{(f_1, f_2, f_3)} = \overline{(\alpha_1 f_1 \alpha_0^{-1}, \alpha_2 f_2 \alpha_1^{-1}, \alpha_3 f_3 \alpha_2^{-1})}.$$

The following identities show that  $\tau$  is well-defined:

$$\begin{aligned} \overline{\tau(\alpha_1 f_1 \alpha_0^{-1}, \alpha_2 f_2 \alpha_1^{-1}, \alpha_3 f_3 \alpha_2^{-1})} &= \overline{((\alpha_2 f_2 f_1 \alpha_0^{-1}, \alpha_3 f_3 \alpha_2^{-1}), (\alpha_1 f_1 \alpha_0^{-1}, \alpha_2 f_2 \alpha_1^{-1}))} = \\ &= \overline{((f_2 f_1, f_3), (f_1, f_2))} = \overline{\tau(f_1, f_2, f_3)}. \end{aligned}$$

To show that  $\tau$  is surjective take a tuple  $(\overline{(f_1, f_2)}, \overline{(f_3, f_4)})$  in  $E$ . We may assume that

$$f_2 f_1 = f \quad \text{and} \quad f_4 f_3 = f_1.$$

So we have that

$$\tau(\overline{f_1, f_2, f_3}) = \overline{((f_1, f_2), (f_3, f_4))}.$$

To show injectivity we proceed as follows. Suppose we are given morphisms

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} x_3 \quad \text{and} \quad y_0 \xrightarrow{g_1} y_1 \xrightarrow{g_2} y_2 \xrightarrow{g_3} y_3,$$

such that

$$\tau(\overline{f_1, f_2, f_3}) = \tau(\overline{g_1, g_2, g_3}) \in \overline{D_3 f}.$$

Then

$$(\overline{(f_2 f_1, f_3)}, \overline{(f_1, f_2)}) = (\overline{(f_2 f_1, f_3)}, \overline{(f_1, f_2)}),$$

and thus we have commutative diagrams with vertical isomorphisms:

$$\begin{array}{ccccccc} x_0 & \xrightarrow{f_1} & x_1 & \xrightarrow{f_2} & x_2 & \xrightarrow{f_2} & x_3 \\ \alpha_0 \downarrow & & & & \alpha_2 \downarrow & & \alpha_3 \downarrow \\ y_0 & \xrightarrow{g_1} & y_1 & \xrightarrow{g_2} & y_2 & \xrightarrow{g_2} & y_3 \end{array}$$
  

$$\begin{array}{ccccc} x_0 & \xrightarrow{f_1} & x_1 & \xrightarrow{f_2} & x_2 \\ \downarrow & & \downarrow & & \downarrow \\ y_0 & \xrightarrow{g_1} & y_1 & \xrightarrow{g_2} & y_2 \end{array}$$

In particular  $x_1 \simeq y_1$  and thus we obtain the isomorphism filling  $\alpha_1$  in the diagram

$$\begin{array}{ccccc} & & x_1 & & \\ & \nearrow f_1 & \downarrow \alpha_1 & \nwarrow f_2 & \\ x_0 & & & & x_2 \\ & \searrow g_1 \alpha_0 & & \nearrow \alpha_2^{-1} g_2 & \\ & & y_1 & & \end{array}$$

So we get the diagram

$$\begin{array}{ccccccc} x_0 & \xrightarrow{f_1} & x_1 & \xrightarrow{f_2} & x_2 & \xrightarrow{f_2} & x_3 \\ \alpha_0 \downarrow & & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow \\ y_0 & \xrightarrow{g_1} & y_1 & \xrightarrow{g_2} & y_2 & \xrightarrow{g_2} & y_3 \end{array}$$

showing injectivity. □

More generally, we have the following result by induction.

**Corollary 96.**

$$\alpha_1 \star \alpha_2 \star \dots \star \alpha_n(\bar{f}) = \sum_{(\overline{f_1, \dots, f_n}) \in \overline{D_n f}} \alpha_1(\bar{f}_1) \alpha_2(\bar{f}_2) \dots \alpha_n(\bar{f}_n).$$

**Corollary 97.** Let  $C$  be an isomorphism filling category with  $\overline{D}_2\overline{f}$  a finite set for  $\overline{f} \in \overline{C}_1$ . The free  $R$ -module  $\langle \overline{C}_1 \rangle$  generated by  $\overline{C}_1$  together with the  $R$ -linear maps

$$\Delta : \langle \overline{C}_1 \rangle \longrightarrow \langle \overline{C}_1 \rangle \otimes \langle \overline{C}_1 \rangle \quad \text{and} \quad \epsilon : \langle \overline{C}_1 \rangle \longrightarrow R$$

given on generators, respectively, by

$$\Delta \overline{f} = \sum_{(\overline{f}_1, \overline{f}_2) \in \overline{D}_2\overline{f}} \overline{f}_1 \otimes \overline{f}_2 \quad \text{and} \quad \epsilon \overline{f} = \begin{cases} 1 & \text{if } f \text{ is an isomorphism,} \\ 0 & \text{if otherwise,} \end{cases}$$

is a  $R$ -coalgebra.

**Theorem 98.** Let  $C$  be an essentially finite decomposition isomorphism filling category. A map  $\alpha \in [\overline{C}_1, R]$  is a  $\star$ -unit if and only if  $\alpha(\overline{1}_x)$  is a unit in  $R$  for all  $x \in C$ . Thus the map  $\xi \in [\overline{C}_1, R]$  constantly equal to 1 is a unit in  $([\overline{C}_1, R], \star)$ ; its inverse  $\mu$  is called the Möbius function of  $C$  and is such that  $\mu \overline{f} = 1$  if  $f$  is an isomorphism, and if  $f$  is not an isomorphism then  $\mu \overline{f}$  is given by

$$\mu \overline{f} = \sum_{n \geq 1} (-1)^n |\overline{PD}_n \overline{f}|.$$

*Proof.* Assume  $\alpha \star \beta = 1$ . Then for all  $x \in C$  we have that

$$\alpha(\overline{1}_x) \beta(\overline{1}_x) = (\alpha \star \beta)(\overline{1}_x) = 1, \quad \text{thus} \quad \alpha(\overline{1}_x) \text{ is a unit.}$$

Conversely, if  $\alpha(\overline{1}_x)$  is a unit for all  $x \in C$ , then  $\alpha$  is invertible and its inverse  $\alpha^{-1}$  is given by

$$\alpha^{-1}(\overline{1}_x) = \frac{1}{\alpha(\overline{1}_x)},$$

and if  $f$  a non-isomorphism then we have that

$$\alpha^{-1}(\overline{f}) = \sum_{n \geq 1} \sum_{(\overline{f}_1, \dots, \overline{f}_n) \in \overline{PD}_n \overline{f}} (-1)^n \frac{\alpha(\overline{f}_1) \dots \alpha(\overline{f}_n)}{\alpha(\overline{1}_{sf_1}) \alpha(\overline{1}_{tf_1}) \dots \alpha(\overline{1}_{tf_n})}.$$

□

**Corollary 99.** Let  $C$  be an essentially finite decomposition isomorphism filling category and let  $R[[x_{\overline{f}}]]$  be the  $R$ -algebra of formal power series in the variables  $x_{\overline{f}}$  for  $\overline{f} \in \overline{C}_1$ , with  $f$  a non-isomorphism in  $C_1$ . The structural maps on  $R[[x_{\overline{f}}]]$  given, respectively, on generators by

$$\epsilon 1 = 1 \quad \text{and} \quad \epsilon x_{\overline{f}} = 0,$$

$$\Delta 1 = 1 \otimes 1 \quad \text{and} \quad \Delta x_f = 1 \otimes x_{\overline{f}} + \sum_{(\overline{f}_1, \overline{f}_2) \in \overline{PD}_2 \overline{f}} x_{\overline{f}_1} \otimes x_{\overline{f}_2} + x_{\overline{f}} \otimes 1,$$

$$S 1 = 1 \quad \text{and} \quad S x_{\overline{f}} = \sum_{n \geq 1} \sum_{(\overline{f}_1, \dots, \overline{f}_n) \in \overline{PD}_n \overline{f}} (-1)^n x_{\overline{f}_1} \dots x_{\overline{f}_n}$$

turn  $R[[x_{\overline{f}}]]$  into a Hopf algebra, and the Möbius function  $\mu \in [C_1, R]$  of  $C$  is given by

$$\mu \overline{f} = Sx_{\overline{f}}(1).$$

.

**Proposition 100.** Let  $C$  and  $D$  be finite essentially finite decomposition isomorphism filling categories, then  $C \times D$  is a finite an essentially finite decomposition isomorphism filling category,

$$\overline{(C \times D)}_1 \simeq \overline{C}_1 \times \overline{D}_1,$$

and we have a natural isomorphism of algebras

$$([\overline{(C \times D)}_1, R], \star) \simeq ([\overline{C}_1, R], \star) \otimes ([\overline{D}_1, R], \star).$$

Moreover, we have that

$$\xi_{C \times D}(\overline{f, g}) = \xi_C \overline{f} \xi_D \overline{g}$$

and thus

$$\mu_{C \times D}(\overline{f, g}) = \mu_C \overline{f} \mu_D \overline{g}.$$

Let  $f : x \longrightarrow y$  be a non-isomorphism in  $C_1$ . We construct the augmented simplicial essentially finite groupoid  $D_* \overline{f}$  which is a full sub-simplicial groupoid of  $C_*(\overline{x}, \overline{y})$ . For  $n \geq -1$ , we have inclusions

$$D_n \overline{f} \subseteq C_n(\overline{x}, \overline{y}) = [[0, n+2], C]_{\overline{x}, \overline{y}}.$$

A functor  $F \in C_n(\overline{x}, \overline{y})$  belongs to  $D_n \overline{f}$  if and only if

$$\overline{F}(0 \leq n+2) = \overline{f}.$$

By Definitions 86 and 87 we have for  $n \geq -1$  that:

$$D_n \overline{f} = D_{n+2} \overline{f} \quad \text{and} \quad D_n \overline{f}^\natural = PD_{n+2} \overline{f}.$$

Taking isomorphisms classes of objects we get the simplicial complex  $\overline{D}_* \overline{f}$  such that

$$\overline{D}_n \overline{f} = \overline{D}_{n+2} \overline{f} \quad \text{and} \quad \overline{D}_n \overline{f}^\natural = \overline{PD}_{n+2} \overline{f}.$$

**Theorem 101.** The Möbius function  $\mu \in [\overline{C}_1, R]$  of a finite decomposition isomorphism filling category  $C$  is given for  $f$  a non-isomorphism by

$$\mu \overline{f} = \tilde{\chi} \overline{D}_* \overline{f}.$$

*Proof.* From Theorem 98 we have that

$$\mu \overline{f} = \sum_{n \geq 1} (-1)^n |\overline{PD}_n \overline{f}| = \sum_{n \geq -1} (-1)^n |\overline{PD}_{n+2} \overline{f}| = \sum_{n \geq -1} (-1)^n |\overline{D}_n \overline{f}^\natural| = \tilde{\chi} \overline{D}_* \overline{f}.$$

□

Note that for  $\bar{x} < \bar{y} \in \bar{C}$  we have the following identity of simplicial groupoids

$$C_*(\bar{x}, \bar{y}) = \bigsqcup_{\bar{f} \in \overline{C(\bar{x}, \bar{y})}} D_* \bar{f}, \quad \text{so we have that}$$

$$\tilde{\chi}_g C_*(\bar{x}, \bar{y}) = \sum_{\bar{f} \in \overline{C(\bar{x}, \bar{y})}} \tilde{\chi}_g D_* \bar{f}.$$

From Theorem 65 we know that the Möbius function  $\mu \in [\bar{C}_1, R]$  of  $C$  is given by

$$\mu[\bar{x}, \bar{y}] = \tilde{\chi}_g C_*(\bar{x}, \bar{y}).$$

Thus we have shown the following result.

**Theorem 102.** Let  $C$  be an essentially finite decomposition isomorphism filling category. Then  $C$  is an essentially locally finite category with Möbius function  $\mu(\bar{x}, \bar{y})$  given for  $\bar{x} < \bar{y}$  in  $\bar{C}$  by

$$\begin{aligned} \mu[\bar{x}, \bar{y}] &= \sum_{\bar{f} \in \overline{C(\bar{x}, \bar{y})}} \tilde{\chi}_g D_* \bar{f}, \quad \text{where} \\ \tilde{\chi}_g D_* \bar{f} &= \sum_{n \geq -1} (-1)^n |D_n \bar{f}^\natural|_g = \sum_{n \geq 1} (-1)^n |\text{PD}_n \bar{f}|_g. \end{aligned}$$

We close this work with an examples illustrating the meaning of Theorem 102.

**Example 103.** Consider the category  $\mathbb{I}$  of finite sets and injective maps. Clearly, we have the identity of posets  $\bar{\mathbb{I}} = \mathbb{N}$ . The incidence map  $\xi \in [\bar{\mathbb{I}}_1, R]$  is given by

$$\xi[n, m] = |\mathbb{I}(\{1, \dots, n\}, \{1, \dots, m\})| = m(m-1)\dots(m-n+1) = \frac{m!}{(m-n)!},$$

and the Möbius function  $\mu \in [\bar{\mathbb{I}}_1, R]$  is given by

$$\mu[n, m] = \frac{(-1)^{m-n}}{n!(m-n)!},$$

since

$$\begin{aligned} \sum_{n \leq k \leq m} \mu[n, k] \xi[k, m] &= \sum_{n \leq k \leq m} \frac{(-1)^{k-n}}{n!(k-n)!} \frac{m!}{(m-k)!} = \\ \binom{m}{n} \sum_{s=0}^{m-n} (-1)^s \binom{m-n}{s} &= \binom{m}{n} (0)^{m-n} = \delta_{n,m}. \end{aligned}$$

The set of equivalence classes of injective maps, i.e equivalence classes of morphisms in  $\mathbb{I}$ , can be identified with the set

$$\{i_{n,m} \mid n \leq m\},$$

where  $i_{n,m}$  is the inclusion map

$$[1, n] = \{1, \dots, n\} \subseteq \{1, \dots, n, \dots, m\} = [1, m].$$

According to Theorem 102 we must have that

$$\mu[n, m] = \tilde{\chi}_g D_* \bar{i}_{n,m} = \sum_{k \geq 1} (-1)^k |\text{PD}_k \bar{i}_{n,m}|_g.$$

The proper  $k$ -decompositions of the morphisms  $i_{n,m}$  are, up to equivalence, given by the inclusions

$$[1, n] \subseteq [1, n_1] \subseteq \dots \subseteq [1, n_{k-1}] \subseteq [1, m],$$

with  $n < n_1 < \dots < n_{k-1} < m$ . The automorphism group of such a decomposition has cardinality

$$\frac{1}{n!(n_1 - n)! \dots (m - n_{k-1})!}.$$

Thus the identity  $\mu[n, m] = \tilde{\chi}_g D_* \bar{i}_{n,m}$  is equivalent to

$$\frac{(-1)^{m-n}}{n!(m-n)!} = \sum_{k \geq 1} \sum_{n < n_1 < \dots < n_{k-1} < m} \frac{(-1)^k}{n!(n_1 - n)!(n_2 - n_1)! \dots (m - n_{k-1})!},$$

and also to

$$(-1)^{m-n} + 1 = \sum_{k \geq 2} \sum_{n < n_1 < \dots < n_{k-1} < m} (-1)^k \binom{m-n}{n, n_1 - n, n_2 - n_1, \dots, m - n_{k-1}}.$$

The latter identity follows directly from Exercise 104 below.

**Exercise 104.** Fix  $a \geq 2$  in  $\mathbb{N}$ . Show by induction that

$$\sum_{k \geq 2} \sum_{a_1 + \dots + a_k = a} (-1)^k \binom{a}{a_1, \dots, a_k} = (-1)^a + 1,$$

where  $a_i \in \mathbb{N}_+$ .

## References

- [1] M. Aguiar, S. Mahajan, Monoidal Functors, Species and Hopf Algebras, Amer. Math. Soc., Providence 2010.
- [2] T. Apostol, Introduction to analytic number theory, Undergraduate Texts in Mathematics, Springer-Verlag, New York 1976.



- [3] J. Baez, J. Dolan, From finite sets to Feynman diagrams, in B. Engquist, W. Schmid (Eds.), *Mathematics unlimited - 2001 and beyond*, Springer, Berlin 2001, pp. 29-50.
- [4] E. Bender, J. Goldman, On the Applications of Möbius Inversion in Combinatorial Analysis, *Amer. Math. Monthly* 82 (1975) 789-803.
- [5] H. Blandín, R. Díaz, Compositional Bernoulli Numbers, *Afric. Diaspora J. Math.* 7 (2009) 119-134.
- [6] H. Blandín, R. Díaz, Rational combinatorics, *Adv. in Appl. Math.* 40 (2008) 107-126.
- [7] E. Castillo, R. Díaz, Rota-Baxter Categories, *Int. Electron. J. Algebra* 5 (2009) 27-57.
- [8] P. Cartier, A primer of Hopf Algebras, in P. Cartier, P. Moussa, B. Julia, P. Vanhove (Eds.), *Frontiers in Number Theory, Physics, and Geometry II*, Springer, Berlin 2007, pp. 537-615.
- [9] P. Cartier, D. Foata, *Problèmes Combinatoires de Commutation et Réarrangements*, *Lect. Notes in Math.* 85, Springer-Verlag, Berlin 1969.
- [10] M. Content, F. Lemay, P. Leroux, Categories de Möbius et fonctorealites: un cadre general pour l'inversion de Möbius. *J. Combin. Theory Ser. A* 28 (1980) 169-190.
- [11] R. Díaz, E. Pariguan, Super, Quantum and Non-Commutative Species, *Afr. Diaspora J. Math.* 8 (2009) 90-130.
- [12] A. Dür, Möbius Functions, Incidence Algebras and Power Series Representations, *Lect. Notes in Math.* 1202, Springer, Berlin 1986.
- [13] T. Fiore, W. Lück, R. Sauer, Euler Characteristic of categories and homotopy colimits, *Doc. Math.* (2011) 301-354.
- [14] P. Goerss, J. Jardine, *Simplicial Homotopy Theory*, Birkhäuser, Basel 2009.
- [15] J. Haigh, On the Möbius algebra and the Grothendieck ring of a finite category, *J. London Math. Soc.* (1980) 81-92.
- [16] A. Joyal, Foncteurs analytiques et espèces de structures, in G. Labelle, P. Leroux (Eds.), *Combinatoire énumérative*, *Lect. Notes in Math.* 1234, Springer, Berlin 1986, pp. 126-159.
- [17] C. Kassel, *Quantum Groups*, Springer, Berlin 1995.
- [18] O. Knill, The Theorems of Grenn-Stokes, Gauss-Bonnet and Poincare-Hopf in Graph Theory, preprint, arXiv:1201.6049.
- [19] D. Kozlov, *Combinatorial Algebraic Topology*, Springer, Berlin 2008.

- [20] F. Lawvere, M. Menni, The Hopf Algebra of Möbius intervals, *Theory and Appl. of Categories* 24 (2010) 221-265.
- [21] F. Lawvere, S. Schanuel, *Conceptual Mathematics*, Cambridge Univ. Press, Cambridge 1997.
- [22] T. Leinster, The Euler characteristic of a category, *Doc. Math.* 13 (2008) 21-49.
- [23] T. Leinster, Notions of Möbius inversion, preprint, arXiv:1201.6049.
- [24] P. Leroux, Les Categories de Möbius, *Cah. Topol. Géom. Différ. Catég.* 16 (1975) 280-282.
- [25] J.-L. Loday, J. Stasheff, A. Voronov (Eds.), *Operads: Proceedings of Renaissance Conferences*, *Contemp. Math.* 202, Amer. Math. Soc., Providence 1997.
- [26] S. Mac Lane, *Categories for the Working Mathematician*, Springer, Berlin 1971.
- [27] J. May, *Symplcial Objects in Algebraic Topology*, Chicago Univ. Press, Chicago 1982.
- [28] J. Kung (Ed.), *Gian-Carlo Rota on Combinatorics*, Birkhäuser, Boston and Basel 1995.
- [29] L. Poinot, G. Duchamp, C. Tollu, Möbius inversion formula for monoids with zero, *Semi-group Forum* 81 (2010) 446-460.
- [30] P. Sarnak, Three Lectures on the Möbius function Ramdonness and Dynamics, [www.math.ias.edu/files/wam/2011/PSMobius.pdf](http://www.math.ias.edu/files/wam/2011/PSMobius.pdf).
- [31] W. Schmitt, Incidence Hopf Algebras, *J. Pure Appl. Algebra* 96 (1994) 299-330.
- [32] T. Tao, The Chowla conjecture and the Sarnak conjecture, <http://terrytao.wordpress.com/tag/mobius-function/>.
- [33] E. Schwab, J. Vallarreal, The Computation of the Möbius Function of a Möbius Category, preprint, arXiv:1210.7697.
- [34] C. Weibel, *An Introduction to Homological Algebra*, Cambridge Univ. Press, Cambridge 1994.

ragadiaz@gmail.com  
 Instituto de Matemáticas y sus Aplicaciones  
 Universidad Sergio Arboleda, Bogotá, Colombia